Today: Partial derivatives

Next: Strang 4.4

Week 5:

- homework 4 (due Friday, October 28)
- regrades of Midterm 1 on Gradescope until October 30
Limit of a function of two variables

**Def.** Consider a point \((a,b)\in \mathbb{R}^2\). A \(\delta\)-disk centered at point \((a,b)\) is the open disk of radius \(\delta\) centered at \((a,b)\)

\[
\{(x,y) \mid (x-a)^2 + (y-b)^2 < \delta^2\}
\]

**Def.** The limit of \(f(x,y)\) as \((x,y)\) approaches \((x_0,y_0)\) is \(L\)

\[
\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L
\]

if for each \(\varepsilon > 0\) there exists a small enough \(\delta > 0\) such that all points in a \(\delta\)-disk around \((x_0,y_0)\), except possible \((x_0,y_0)\) itself, \(f(x,y)\) is no more than \(\varepsilon\) away from \(L\). (For any \(\varepsilon > 0\) there exists \(\delta > 0\) such that \(|f(x,y) - L| < \varepsilon\) whenever \(\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta\).)
Limit of a function of two variables

This definition ensures that if \( \lim_{(x,y) \to (x_0,y_0)} f(x,y) = L \), then any way of approaching \((x_0,y_0)\) results in the same limit \(L\).

(Another) example when the limit fails to exist:

\[
\lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4} \text{ does not exist}
\]

- approach \((0,0)\) along the line \(x = 0\); on this line \(\frac{0 \cdot y^2}{0^2 + y^4} = 0\)
- approach \((0,0)\) along the curve \(x = y^2\), \(\frac{y^2 \cdot y^2}{y^4 + y^4} = \frac{1}{2}\)
Computing limits. Limit laws

**Theorem 4.1** Let \( \lim_{(x,y) \to (a,b)} f(x,y) = L \), \( \lim_{(x,y) \to (a,b)} g(x,y) = M \), \( c - constant \).

- \( \lim_{(x,y) \to (a,b)} c = c \)
- \( \lim_{(x,y) \to (a,b)} x = a \)
- \( \lim_{(x,y) \to (a,b)} y = b \)

- \( \lim_{(x,y) \to (a,b)} [f(x,y) + g(x,y)] = L + M \)
- \( \lim_{(x,y) \to (a,b)} [f(x,y)g(x,y)] = LM \)

- If \( M \neq 0 \), \( \lim_{(x,y) \to (a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M} \)

- \( \lim_{(x,y) \to (a,b)} [c \cdot f(x,y)] = cL \)
- \( \lim_{(x,y) \to (a,b)} [f(x,y)]^n = L^n \)
- \( \lim_{(x,y) \to (a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} \)
Examples

\[ \lim_{(x,y) \to (1,2)} \frac{\sqrt{3x-y}}{(x^2 + xy + y^3)^2} = \lim_{(x,y) \to (1,2)} \frac{\sqrt{\lim_{(x,y) \to (1,2)} [3x-y]}}{( \lim_{(x,y) \to (1,2)} [x^2 + xy + y^3] )^2} \]

\[ = \frac{\left( \lim_{(x,y) \to (1,2)} x \right)^2 + \left( \lim_{(x,y) \to (1,2)} x \right) \left( \lim_{(x,y) \to (1,2)} y \right) + \left( \lim_{(x,y) \to (1,2)} y^3 \right)^2}{\sqrt{3 \lim_{(x,y) \to (1,2)} x - \lim_{(x,y) \to (1,2)} y}} \]

\[ = \frac{\sqrt{3 \cdot 1 - 2}}{(1^2 + 1 \cdot 2 + 2^3)^2} = \frac{1}{121} \]
Continuity of functions of two variables

Def. A function $f(x, y)$ is continuous at a point $(a, b)$ if

(i) $f(a, b)$ exists;

(ii) $\lim_{(x,y) \to (a,b)} f(x, y)$ exists; and

(iii) $\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)$

Properties

1. If $f(x, y)$ and $g(x, y)$ are continuous at $(x_0, y_0)$, then $f(x, y) \pm g(x, y)$ is continuous at $(x_0, y_0)$

2. If $\psi(x)$ is continuous at $x_0$ and $\psi(y)$ is continuous at $y_0$, then $f(x, y) = \psi(x) \psi(y)$ is continuous at $(x_0, y_0)$
Continuity of functions of two variables

Properties (cont.)

3. If $g(x,y)$ is continuous at $(x_0,y_0)$, and $f(z)$ is continuous at $z_0 := g(x_0,y_0)$, then

$$f \circ g (x,y) = f(g(x,y))$$

is continuous at $(x_0,y_0)$. 
Continuity of functions of two variables

Example: \( \frac{\sqrt{3x-y}}{(x^2+xy+y^3)^2} \): 3x-y is continuous on \( \mathbb{R}^2 \)

\( f(g(x,y)) = f_i(x,y) \)

\( f(z) = \sqrt{z} \) is continuous for all \( z \geq 0 \)

so \( \sqrt{3x-y} \) is continuous for all \((x,y)\) such that \( 3x-y \geq 0 \)

Similarly, \( g(x,y) = x^2 + xy + y^3 \) is continuous on \( \mathbb{R}^2 \)

\( f(z) = \frac{1}{z^2} \) is continuous for all \( z \neq 0 \)

\( f(g(x,y)) = \frac{1}{(x^2+xy+y^3)^2} \) is continuous at all \((x,y)\) such that \( x^2 + xy + y^3 \neq 0 \)

Take \((x_0,y_0) = (1,2)\). Then \( 3 \cdot 1 - 2 = 1 > 0 \), \( 1^2 + 1 \cdot 2 + 2^3 = 11 \neq 0 \), so both \( f_1 \) and \( f_2 \) are continuous at \((1,2)\) and thus

\( \lim_{(x,y) \to (1,2)} f_1(x,y) f_2(x,y) = \lim_{(x,y) \to (1,2)} f_i(x,y) \lim_{(x,y) \to (1,2)} f_2(x,y) = f_1(1,2) f_2(1,2) = 1 \cdot \frac{1}{12} = \frac{1}{12} \)