MATH 10C: Calculus III (Lecture B00)

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Today: Projectile motion. Functions of two variables

Next: Strang 4.1

Week 5:

- homework 4 (due Friday, October 27)
- regrades of Midterm 1 on Gradescope until October 29
Properties of derivatives of vector-valued functions

Thm 3.3. Let \( \mathbf{r}(t) \) and \( \mathbf{u}(t) \) be differentiable vector-valued functions, let \( f(t) \) be a differentiable scalar function, let \( c \) be a scalar.

(i) \[ \frac{d}{dt} [c \mathbf{r}(t)] = \] (scalar multiple)

(ii) \[ \frac{d}{dt} [ \mathbf{r}(t) \pm \mathbf{u}(t)] = \] (sum and difference)

(iii) \[ \frac{d}{dt} [ f(t) \mathbf{r}(t)] = \] (product with scalar function)

(iv) \[ \frac{d}{dt} [ \mathbf{r}(t) \cdot \mathbf{u}(t)] = \] (dot product)

(v) \[ \frac{d}{dt} [ \mathbf{r}(t) \times \mathbf{u}(t)] = \] (cross product)

(vi) \[ \frac{d}{dt} [ \mathbf{r}(f(t))] = \] (chain rule)
Properties of derivatives of vector-valued functions

(vii) If \( \vec{r}(t) \cdot \vec{r}(t) = c \), then

Proof (iv) \[
\frac{d}{dt} \left[ \vec{r}(t) \cdot \vec{u}(t) \right]
= \]
= \]
= \]
= \]

(vii) \[
\frac{d}{dt} \left[ \vec{r}(t) \cdot \vec{r}(t) \right]
\]

This means that if \( \| \vec{r}(t) \| \) is constant, then
Motion in space

If \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) is the position of the particle at time \( t \), then

- \( \mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle \) is the velocity, and
- \( \mathbf{a}(t) = \mathbf{r}''(t) = \langle x''(t), y''(t), z''(t) \rangle \) is the acceleration, and
- \( v(t) = \| \mathbf{v}(t) \| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \) is the speed.

Example: Projectile motion

Consider an object moving with initial velocity \( \mathbf{v}_0 \) but with no forces acting on it other than gravity (ignore the effect of air resistance).

Newton's second law: 

Earth's gravity: 

where \( m = \) mass of the object and \( g = 9.8 \text{ m/s}^2 \)
Projectile motion

Fix the coordinate system: $\mathbf{g} = y_a$.

Backward $\rightarrow \mathbf{F}_g$.

Newton's second law: $\mathbf{F} = m\mathbf{a}$.

Earth's gravity is the only force acting on the object.
Projectile motion

\[ \vec{F}(t) = \vec{F}_g : \] (constant acceleration)

Since \( \vec{a}(t) = \vec{v}'(t) \), we have

Take antiderivative: \( \vec{v}(t) = \) (initial velocity):

Determine \( \vec{c}_1 \) by taking

\[ \vec{v}(o) = \vec{v}(o) = \]

This gives the velocity of the object:

\[ \vec{v}(t) = \]

Similarly, \( \vec{v}(t) = \) . By taking the antiderivative and \( \vec{r}(0) = \vec{r}_0 \),

\[ \vec{r}(t) = \int \vec{v}(t) \, dt + \vec{c}_0 = \]

\[ \vec{r}(o) = \vec{c}_0 = \vec{r}_0 , \text{ so } \]
Projectile motion

A projectile is shot by a howitzer with initial speed 800 m/s on a flat terrain. Determine the max distance the projectile can cover before hitting the ground.

Since the initial speed is given, the initial velocity can be determined by the angle:

Equation of the trajectory: \( \vec{r}(t) = \)

Hitting the ground: second component or \( \vec{r}(t) \) is 0:

\[ \vec{r}(t_n) = \]

so

\[ \vec{r}(t_n) = \]

when

\[ \text{i.e.,} \]

The position of the hit is

Maximum is achieved

Max distance is 32 km.
Functions of several variables
Functions of two variables

Def. A function of two variables maps each element in a subset $D \subset \mathbb{R}^2$ to a real number. The set $D$ is called the domain of the function. The range of $f$ is the set of all real numbers $z$ that has at least one ordered pair $(x,y) \in D$ s.t. $f(x,y) = z$.

If not specified, we choose the domain to be the set of all pairs $(x,y)$ for which $f(x,y)$ is well-defined.
Functions of two variables

Example. Find the domain and range of the function

\[ f(x, y) = \sqrt{4 - x^2 - y^2} \]

Domain. One restriction: the number under the square root has to be nonnegative, i.e.,

The set of all pairs \((x, y) \in \mathbb{R}^2\) such that

\[ x^2 + y^2 \leq 4 \]

is a

The domain of \( f \) is

Range. For \((x, y)\) in the domain the range of \( x^2 + y^2 \) is

the range of \( 4 - x^2 - y^2 \) is

the range of \( \sqrt{4 - x^2 - y^2} \) is
Graph of a function of two variables

Function f of two variables: maps each pair \((x,y)\) from its domain to a real number \(z = f(x,y)\).

The graph of \(f\) consists of ordered triples \((x,y, f(x,y))\) for all \((x,y)\) in the domain of \(f\). We call the graph of a function of two variables a surface.

Example \(f(x,y) = \sqrt{4-x^2-y^2}\), \(\text{dom}(f) = \{(x,y) \mid x^2+y^2 \leq 4\}\)

Graph of \(f\) consists of all \((x,y,z)\in \mathbb{R}^3\) such that \(z = \sqrt{4-x^2-y^2}\), or

- equation of a
Level curves

Def. Given a function \( f(x, y) \) and a number \( c \) in the range of \( f \), a level curve of a function of two variables for the value \( c \) is defined to be

Example \( f(x, y) = \sqrt{4 - x^2 - y^2} \)

Range of \( f \) is \([0, 2]\).

Take . Then the level curve of \( f \) for is defined by
Functions of more than two variables

In a similar way we can define functions of more than two variables, e.g., functions of three variables:

\[ f(x, y, z) = \]  

such that, i.e.