Today: Projectile motion. Functions of two variables

Next: Strang 4.1

Week 5:

- homework 4 (due Friday, October 27)
- regrades of Midterm 1 on Gradescope until October 29
Properties of derivatives of vector-valued functions

Thm 3.3. Let \( \mathbf{r}(t) \) and \( \mathbf{u}(t) \) be differentiable vector-valued functions, let \( f(t) \) be a differentiable scalar function, let \( c \) be a scalar.

(i) \[ \frac{d}{dt} [c \mathbf{r}(t)] = c \mathbf{r}'(t) \] (scalar multiple)

(ii) \[ \frac{d}{dt} [\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t) \] (sum and difference)

(iii) \[ \frac{d}{dt} [f(t) \mathbf{r}(t)] = f'(t) \mathbf{r}(t) + f(t) \mathbf{r}'(t) \] (product with scalar function)

(iv) \[ \frac{d}{dt} [\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}'(t) \cdot \mathbf{u}(t) + \mathbf{r}(t) \cdot \mathbf{u}'(t) \] (dot product)

(v) \[ \frac{d}{dt} [\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}'(t) \times \mathbf{u}(t) + \mathbf{r}(t) \times \mathbf{u}'(t) \] (cross product)

(vi) \[ \frac{d}{dt} [\mathbf{r}(f(t))] = \mathbf{r}'(f(t)) \cdot f'(t) \] (chain rule)
Properties of derivatives of vector-valued functions

(vii) If \( \mathbf{r}(t) \cdot \mathbf{r}(t) = c \), then \( \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \)

**Proof** (iv) \( \frac{d}{dt} [ \mathbf{r}(t) \cdot \mathbf{u}(t) ] \)

\[
\frac{d}{dt} \left[ \mathbf{r}_1(t) u_1(t) + \mathbf{r}_2(t) u_2(t) + \mathbf{r}_3(t) u_3(t) \right] = \frac{d}{dt} \left[ \mathbf{r}_1'(t) u_1(t) + \mathbf{r}_2'(t) u_2(t) + \mathbf{r}_3'(t) u_3(t) \right] = \mathbf{r}'(t) \cdot \mathbf{u}(t) + \mathbf{r}(t) \cdot \mathbf{u}'(t) \\
= \mathbf{r}'(t) \cdot \mathbf{u}'(t) + \mathbf{r}(t) \cdot \mathbf{u}(t) \]

(vii) \( 0 = \frac{d}{dt} [ \mathbf{r}(t) \cdot \mathbf{r}(t) ] = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2 \mathbf{r}(t) \cdot \mathbf{r}'(t) \)

This means that if \( \| \mathbf{r}(t) \| \) is constant, then \( \mathbf{r}(t) \perp \mathbf{r}'(t) \)
Motion in space

If \( \vec{r}(t) = (x(t), y(t), z(t)) \) is the position of the particle at time \( t \), then

- \( \vec{v}(t) = \vec{r}'(t) = (x'(t), y'(t), z'(t)) \) is the velocity, and
- \( \vec{a}(t) = \vec{r}''(t) = (x''(t), y''(t), z''(t)) \) is the acceleration, and
- \( v(t) = \| \vec{v}(t) \| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \) is the speed.

Example: Projectile motion

Consider an object moving with initial velocity \( \vec{v}_0 \) but with no forces acting on it other than gravity (ignore the effect of air resistance).

Newton’s second law: \( \vec{F} = m \vec{a} \), where \( m \) = mass of the object

Earth’s gravity: \( \| \vec{F}_g \| = m \cdot g \), where \( g \approx 9.8 \text{ m/s}^2 \)
Projectile motion

Fix the coordinate system:

\[ \vec{F}_g = -mg \hat{j} = \langle 0, -mg \rangle \]

Newton's second law: \[ \vec{F} = ma \]

Earth's gravity: \[ \vec{F}_g = -mg \hat{j} \]

Earth's gravity is the only force acting on the object: \[ \vec{F} = \vec{F}_g \]
Projectile motion

\[ \vec{F}(t) = \vec{F}_g : \quad m \vec{a}(t) = -mg \cdot \hat{j} \]

\[ \vec{a}(t) = -g \cdot \hat{j} \quad (\text{constant acceleration}) \]

Since \( \vec{a}(t) = \vec{v}'(t) \), we have \( \vec{v}'(t) = -g \cdot \hat{j} \).

Take antiderivative:

\[ \vec{v}(t) = \int -g \cdot \hat{j} \, dt = -gt \cdot \hat{j} + \vec{c}_1 \]

Determine \( \vec{c}_1 \) by taking \( \vec{v}(0) = \vec{v}_o \) (initial velocity):

\[ \vec{v}(0) = -g \cdot 0 + \vec{c}_1 = \vec{c}_1 = \vec{v}_o \]

This gives the velocity of the object:

\[ \vec{v}(t) = -gt \cdot \hat{j} + \vec{v}_o \]

Similarly, \( \vec{v}(t) = \vec{r}'(t) \). By taking the antiderivative and \( \vec{r}(0) = \vec{r}_o \),

\[ \vec{r}(t) = \int \vec{v}(t) \, dt = -g \frac{t^2}{2} \cdot \hat{j} + t \cdot \vec{v}_o + \vec{c}_0 \]

\[ \vec{r}(0) = \vec{c}_0 = \vec{r}_o, \quad \text{so} \quad \vec{r}(t) = -g \frac{t^2}{2} + t \vec{v}_o + \vec{r}_o \]
Projectile motion

A projectile is shot by a howitzer with initial speed 800 m/s on a flat terrain. Determine the max distance the projectile can cover before hitting the ground.

Equation of the trajectory: \( \vec{r}(t) = -10 \cdot \frac{t^2}{2} \cdot \hat{j} + 800 \cdot t \cdot \cos \theta \cdot \hat{i} + 800 \cdot t \cdot \sin \theta \cdot \hat{j} \)

Hitting the ground: second component or \( \vec{r}(t) \) is 0: \(-5t^2 + 800 \cdot t \cdot \sin \theta = 0 \)
\( t(-5t + 800 \sin \theta) = 0 \), so \( t_h = \frac{800 \sin \theta}{5} = 160 \cdot \sin \theta \). The position of the hit is \( \vec{r}(t_h) = 0 \cdot \hat{j} + 800 \cdot 160 \cdot \sin \theta \cdot \cos \theta \cdot \hat{i} = 64000 \cdot \sin(2\theta) \). Maximum is achieved when \( \sin(2\theta) = 1 \), i.e., \( 2\theta = \frac{\pi}{2} \), \( \theta = \frac{\pi}{4} = 45^\circ \). Max distance is 64 km.
Functions of several variables
**Functions of two variables**

**Def.** A function of two variables maps each ordered pair \((x,y)\) in a subset \(D \subseteq \mathbb{R}^2\) to a unique real number \(z = f(x,y)\). The set \(D\) is called the domain of the function. The range of \(f\) is the set of all real numbers \(z\) that has at least one ordered pair \((x,y) \in D\) s.t. \(f(x,y) = z\).

If not specified, we choose the domain to be the set of all pairs \((x,y)\) for which \(f(x,y)\) is well-defined.