Today: Calculus of vector-valued functions. Tangent lines

Next: Strang 3.4

Week 4:

- homework 4 (due Friday, October 27)
Derivatives of vector-valued functions

The derivative of a vector-valued function \( \mathbf{r} \) is

\[
\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}
\]

provided that the limit exists. If \( \mathbf{r}'(t) \) exists, we say that \( \mathbf{r} \) is differentiable at \( t \). If \( \mathbf{r} \) is differentiable at every point \( t \) from the interval \( (a,b) \), we say that \( \mathbf{r} \) is differentiable on \( (a,b) \).

Notice that if \( \mathbf{r}(t) = \langle r_1(t), r_2(t), r_3(t) \rangle \), then

\[
\mathbf{r}'(t) = \lim_{h \to 0} \frac{\langle r_1(t+h)-r_1(t), r_2(t+h)-r_2(t), r_3(t+h)-r_3(t) \rangle}{h}
\]

\[
= \langle \lim_{h \to 0} \frac{r_1(t+h)-r_1(t)}{h}, \lim_{h \to 0} \frac{r_2(t+h)-r_2(t)}{h}, \lim_{h \to 0} \frac{r_3(t+h)-r_3(t)}{h} \rangle = \langle r_1'(t), r_2'(t), r_3'(t) \rangle
\]
Calculus of vector-valued functions

Example  Let \( \vec{r}(t) = \langle \sin t, e^{2t}, t^2-4t+2 \rangle \)

Then \( \vec{r}'(t) = \langle \cos t, 2e^{2t}, 2t-4 \rangle \)

Summary

Calculus concepts (limit, continuity, derivative) are applied to vector-valued functions componentwise (apply to each component separately).

If \( \vec{r}(t) \) represents the position of some object, then

- \( \vec{r}'(t) \) is the velocity of this object (\( \| \vec{r}'(t) \| \) is speed)
- \( \vec{r}''(t) \) is the acceleration of the object
Tangent vectors. Tangent lines

Let $\vec{r}(t)$ be a vector-valued function. Suppose that $\vec{r}$ is differentiable at $t_0$. Let $C$ be a curve defined (parametrized) by $\vec{r}(t)$.

Then vector $\vec{r}'(t)$ is tangent to $C$ at $t_0$ (at $\vec{r}(t_0)$).

The tangent line to $\vec{r}$ at $t_0$ is the line given by the vector equation $\vec{L}(t) = \vec{r}(t_0) + \vec{r}'(t_0)(t - t_0)$. 

$$\vec{r}(t) = \langle \cos t, \sin t \rangle$$
$$\vec{r}'(t) = \langle -\sin t, \cos t \rangle$$
$$\vec{r}'(\frac{\pi}{4}) = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$$
Tangent vectors. Tangent lines

The tangent line \( \vec{L}(t) \) to \( \vec{r}(t) \) at \( t_0 \) has the same position and velocity as \( \vec{r} \) at time \( t_0 \):

\[
\vec{L}(t_0) = \vec{r}(t_0)
\]

\[
\vec{L}'(t_0) = \vec{r}'(t_0)
\]

Example: Imagine a satellite orbiting a planet.

If the planet disappears at time \( t_0 \), then the satellite would keep going along \( \vec{L}(t) \).
Tangent vectors. Tangent lines

Example  Let \( \vec{r}(t) = \langle t^2 - 2, e^{3t}, t \rangle \)

Find the tangent line to \( \vec{r}(t) \) at \( t_0 = 1 \).

First, find the tangent vector at \( t_0 = 1 \)

\[
\vec{r}'(t) = \langle 2t, 3e^{3t}, 1 \rangle
\]

\[
\vec{r}'(1) = \langle 2, 3e^3, 1 \rangle
\]

Next, find the position at \( t_0 = 1 \)

\[
\vec{r}(1) = \langle -1, e^3, 1 \rangle
\]

Finally, we can write the equation for the tangent line

\[
\vec{e}(t) = \langle -1, e^3, 1 \rangle + \langle 2, 3e^3, 1 \rangle \cdot (t - 1)
\]

Definition  We call \( T(t) := \frac{\vec{r}'(t)}{||\vec{r}'(t)||} \) the principal unit tangent vector to \( \vec{r} \) at \( t \) (provided \( ||\vec{r}'(t)|| \neq 0 \)).
Integrals of vector-valued functions

Integration of vector-valued functions is done componentwise: if \( \mathbf{r}(t) = \langle r_1(t), r_2(t), r_3(t) \rangle \), then

\[
\int \mathbf{r}(t) \, dt = \langle \int r_1(t) \, dt, \int r_2(t) \, dt, \int r_3(t) \, dt \rangle \quad \text{(antiderivative)}
\]

and if \( a < b \)

\[
\int_a^b \mathbf{r}(t) \, dt = \langle \int_a^b r_1(t) \, dt, \int_a^b r_2(t) \, dt, \int_a^b r_3(t) \, dt \rangle \quad \text{(definite integral)}
\]

Example
\[
\int \langle \sin t, t^2 + 2t, e^{2t} \rangle \, dt = \langle -\cos t + c_1, \frac{t^3}{3} + t^2 + c_2, \frac{e^{2t}}{2} + c_3 \rangle
\]

\[
= \langle -\cos t, \frac{t^3}{3} + t^2, \frac{e^{2t}}{2} \rangle + \mathbf{C}, \quad \text{where } \mathbf{C} \text{ is an arbitrary vector of constants}
\]

\[
\frac{2}{3} \int_0^2 \langle \sin t + (t^2 + 2t) \, \mathbf{j} + e^{2t} \, \mathbf{k} \rangle \, dt = \langle -\cos(z) + \cos(0), \frac{2^3}{3} + 2^2 - 0, \frac{e^{2 \cdot 2}}{2} - \frac{e^{2 \cdot 0}}{2} \rangle
\]

\[
= \langle -\cos(z) + 1, \frac{8}{3} + 4, \frac{e^4}{2} - \frac{1}{2} \rangle
\]
Integrals of vector-valued functions

Fundamental theorem of calculus

Let \( \mathbf{f} : [a, b] \to \mathbb{R}^3 \) be a continuous vector-valued function.

Let \( \mathbf{F} : [a, b] \to \mathbb{R}^3 \) be such that \( \mathbf{F}' = \mathbf{f} \) (\( \mathbf{F} \) is antiderivative of \( \mathbf{f} \)). Then

\[
\int_a^b \mathbf{f}(t) \, dt = \mathbf{F}(b) - \mathbf{F}(a)
\]

In particular,

- if \( \mathbf{V}(t) \) is the velocity vector, \( \mathbf{r}(t) \) is the position, then

\[
\int_a^b \mathbf{V}(t) \, dt = \mathbf{r}(b) - \mathbf{r}(a)
\]

...gives the displacement between times \( a \) and \( b \)

- if \( \mathbf{a}(t) \) is the acceleration, then

\[
\int_a^b \mathbf{a}(t) \, dt = \mathbf{V}(b) - \mathbf{V}(a)
\]
Properties of derivatives of vector-valued functions

Thm 3.3. Let \( \mathbf{r}(t) \) and \( \mathbf{u}(t) \) be differentiable vector-valued functions, let \( f(t) \) be a differentiable scalar function, let \( c \) be a scalar.

(i) \[ \frac{d}{dt}[c \mathbf{r}(t)] = c \mathbf{r}'(t) \] (scalar multiple)

(ii) \[ \frac{d}{dt}[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t) \] (sum and difference)

(iii) \[ \frac{d}{dt}[f(t) \mathbf{r}(t)] = f'(t)\mathbf{r}(t) + f(t) \cdot \mathbf{r}'(t) \] (product with scalar function)

(iv) \[ \frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}'(t) \cdot \mathbf{u} + \mathbf{r} \cdot \mathbf{u}' \] (dot product)

(v) \[ \frac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}'(t) \times \mathbf{u}(t) + \mathbf{r}(t) \times \mathbf{u}'(t) \] (cross product)

(vi) \[ \frac{d}{dt}[\mathbf{r}(f(t))] = \mathbf{r}'(f(t)) \cdot f'(t) \] (chain rule)
Properties of derivatives of vector-valued functions

(vii) If \( \mathbf{r}(t) \cdot \mathbf{r}(t) = c \), then \( \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \)

Proof: (iv) \[ \frac{d}{dt} [ \mathbf{r}(t) \cdot \mathbf{u}(t) ] \]

This means that if \( \| \mathbf{r}(t) \| \) is constant, then