MATH 10C: Calculus III (Lecture B00)

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Today: Vector-valued functions

Next: Strang 3.2

Week 3:

- homework 3 (due Tuesday, October 18)
- Midterm 1: Wednesday, October 19 (vectors, dot product, cross product, equations of lines and planes)
Consider a plane containing point \( P = (x_0, y_0, z_0) \) with normal vector \( \vec{n} = \langle a, b, c \rangle \). Then point \( X = (x, y, z) \) belong to this plane if and only if
\[
\vec{n} \perp \overrightarrow{PX}, \text{ i.e. } \vec{n} \cdot \overrightarrow{PX} = 0
\]
vector equation of a plane

\[(*) \quad a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \]
scalar equation of a plane

If we denote \( d := -ax_0 - by_0 - cz_0 \), then \((*)\) becomes
\[
a x + b y + c z + d = 0
\]
general form of the equation of a plane
**Parallel and intersecting planes**

Let $P_1$ and $P_2$ be two planes in $\mathbb{R}^3$. Then the following possibilities exist:

<table>
<thead>
<tr>
<th>$P_1$ and $P_2$ share a common point</th>
<th>YES</th>
<th>NO</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Normal vectors</strong></td>
<td>YES</td>
<td>Equal</td>
</tr>
<tr>
<td>of $P_1$ and $P_2$ are parallel</td>
<td>NO</td>
<td>Intersecting</td>
</tr>
</tbody>
</table>

If two planes intersect, the intersection is a line!
Finding the line of intersection for two planes

Find the parametric and symmetric equations for the line formed by the intersection of the planes

\[ x + 2y + 3z = 0, \quad x + y + z + 1 = 0 \]

\[ \mathbf{n}_1 = \langle 1, 2, 3 \rangle, \quad \mathbf{n}_2 = \langle 1, 1, 1 \rangle \]

\[ k \cdot \mathbf{n}_2 = \mathbf{n}_1 \] does not exist

such \( k \) does not exist

so \( \mathbf{n}_1 \) is not parallel to \( \mathbf{n}_2 \)

(1) \[ \begin{cases} x + 2y + 3z = 0 \\ y + 2z = 1 \end{cases} \]

(2) \[ \begin{cases} x + y + z = -1 \\ y = -2z + 1 \end{cases} \]

Take \( z = t \). Then \( y = -2t + 1 \). Substitute \( z = t \) and \( y = -2t + 1 \) into (1) or (2)

\[ \begin{cases} x + (-2t + 1) + t = -1 \\ x = t - 2 \end{cases} \] parametric

\[ \begin{cases} x = t - 2 \\ y = -2t + 1 \end{cases} \] eq. of a line

\[ \begin{cases} z = t \end{cases} \]

\[ \begin{cases} x + 2 = \frac{y - 1}{-2} = \frac{z}{t} \\ \frac{x - 2}{t} = \frac{y - 1}{-2} = \frac{z}{t} \end{cases} \] symmetric eq. of the same line
**Vector-valued functions**

**Definition** A vector-valued function is a function that takes real numbers as inputs and gives vectors as outputs, i.e.,

\[ \vec{r}(t) = \langle f(t), g(t) \rangle \] - function from \( \mathbb{R} \) to \( \mathbb{R}^2 \)

\[ \vec{r}(t) = \langle f(t), g(t), h(t) \rangle \] - function from \( \mathbb{R} \) to \( \mathbb{R}^3 \)

**Example**

\[ \vec{r}(t) = \langle \cos t, \sin t \rangle \]

\[ \vec{r}(t) = 2t \hat{i} - e^t \hat{j} + 0 \hat{k} = \langle 2t, -e^t, 0 \rangle \]

**Remark** From now on we will not distinguish between the point \((x, y, z)\) and the vector \(\langle x, y, z \rangle\), both are just lists of three real numbers.
Vector-valued functions

Vector valued function \( \vec{F}(t) \) often represents a vector or a position in the space at time \( t \).

Think about the motion of a planet, flight of an airplane or a bird etc.

A vector-valued function may not be defined for all real numbers. For example, \( \vec{F}(t) = \left( \frac{1}{t}, \frac{1}{\text{cost}}, t \right) \) is not defined for \( t = 0 \), and \( t = \frac{\Pi}{2} + \Pi n \), \( n \) is an integer.

You can explicitly specify the set of real numbers for which you want to define the function by writing, e.g., \( \vec{F} : [0,1] \rightarrow \mathbb{R}^3 \). We call this set the domain of \( \vec{F} \).
Vector-valued functions

If the domain is not explicitly specified, we assume that it is the set of all real numbers for which all (three) components of \( \vec{r} \) are defined.

Example

\[
\vec{r}(t) = \left< \frac{1}{t}, \frac{1}{\cos t}, t \right>
\]

\[
\text{dom}(\vec{r}(t)) = \{ t \mid t \neq 0 \text{ and } t \neq \frac{\pi}{2} + \pi n, n \text{ integer} \}
\]

Sometimes the domain is found from the problem setup. If the function describes the motion of a bird between time 0 and time T, then the domain is the interval \([0, T]\).