Name (last, first): ____________________

Student ID: ____________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You may assume that all transition probability functions are STATIONARY.**

☐ **You are allowed to use one 8.5 by 11 inch sheet of paper with handwritten notes (on both sides); no other notes (or books) are allowed.**

1. (40 points) Let $Y \sim \text{Exp}(\lambda)$ for $\lambda > 0$, and let $X \sim \text{Unif}([0, Y])$.
   (a) Compute $E(X)$.
   (b) Show that
$$\int_0^\infty y^2 \lambda e^{-\lambda y} dy = \frac{2}{\lambda^2}.$$
   (c) Use part (b) to compute $E(X^2)$.
   (d) Compute the variance $\text{Var}(X)$ of the random variable X.

Solution.

(a) We compute $E(X)$ by conditioning on the value of Y
$$E(X) = \int_0^\infty E(X|Y = y)\lambda e^{-\lambda y} dy. \quad (1)$$
Since $X \sim \text{Unif}([0, Y])$, we find that $E(X|Y = y) = \frac{y}{2}$. Thus,
$$E(X) = \int_0^\infty \frac{y}{2}\lambda e^{-\lambda y} dy = \frac{1}{2}E(Y) = \frac{1}{2\lambda}. \quad (2)$$

(b) Using integration by parts we have
$$\int_0^\infty y^2 \lambda e^{-\lambda y} dy = -\int_0^\infty y^2 d\big(e^{-\lambda y}\big) = -y^2 e^{-\lambda y}\Big|_0^\infty + \int_0^\infty 2y e^{-\lambda y} dy = \frac{2}{\lambda}E(Y) = \frac{2}{\lambda^2}. \quad (3)$$

(c) We compute $E(X^2)$ by conditioning again on the value of Y
$$E(X^2) = \int_0^\infty E(X^2|Y = y)\lambda e^{-\lambda y} dy = \int_0^\infty \frac{y^2}{3}\lambda e^{-\lambda y} dy = \frac{2}{3\lambda^2}, \quad (4)$$
where on the second step we used that $E(X^2|Y = y) = \frac{y^2}{3}$, and on the last step we used part (b).

(d) Finally,
$$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{2}{3\lambda^2} - \frac{1}{4\lambda^2} = \left(\frac{2}{3} - \frac{1}{4}\right)\frac{1}{\lambda^2} = \frac{5}{12\lambda^2}. \quad (5)$$

2. (30 points) The time intervals between two consecutive rainstorms in San Diego are independent identically distributed random variables with density (in years)

$$f(x) = \begin{cases} 2x, & x \in (0,1) \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

(a) Compute the long run expected time between the last rainstorm and the next rainstorm.

(b) What is the long run probability that it has been at most 6 months since the last rainstorm?

Solution.

(a) If $\delta(t)$ is the current life (age) of the renewal process at time t (time from the last rainstorm to time t), and $\gamma(t)$ is the residual life of the renewal process at time t (time until the next rainstorm after time t), then we have to compute

$$\lim_{t\to\infty} E(\delta(t) + \gamma(t)) = \lim_{t\to\infty} E(\beta(t)). \tag{7}$$

Lecture 18, page 6:

$$\lim_{t\to\infty} E(\beta(t)) = \frac{\sigma^2 + \mu^2}{\mu}, \tag{8}$$

where μ and σ^2 are the mean and variance of the interrenewal times.

$$\mu = \int_0^1 2x^2 dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}, \tag{9}$$

$$\mu^2 + \sigma^2 = \int_0^1 2x^3 dx = \frac{x^4}{2}\Big|_0^1 = \frac{1}{2}, \tag{10}$$

therefore

$$\lim_{t\to\infty} E(\beta(t)) = \frac{3}{4}. \tag{11}$$

(b) In terms of the renewal process, the long run probability that it has been at most 6 months since the last rainstorm is given by

$$\lim_{t\to\infty} P(\delta(t) < 1/2). \tag{12}$$

Lecture 17, page 4:

$$\lim_{t\to\infty} P(\delta(t) < 1/2) = \int_0^{1/2} \frac{1}{\mu}(1 - F(x))dx, \tag{13}$$

where $F(x)$ is the interrenewal distribution. Note, that $F(x) = 1$ for $x \geq 1$. For $x \in (0,1)$

$$F(x) = \int_0^x 2s ds = x^2. \tag{14}$$

Therefore,

$$\lim_{t\to\infty} P(\delta(t) < 1/2) = \int_0^{1/2} \frac{3}{2}(1 - x^2)dx = \frac{3}{2}\Big(x - \frac{x^3}{3}\Big)\Big|_0^{1/2} = \frac{11}{16}. \tag{15}$$

3. (30 points) Suppose that a certain company is using age replacement policy for replacing lightbulbs in its offices: a lightbulb is replaced either upon its failure, or after reaching age $T > 0$, whichever comes first. Suppose that each replacement costs 1 dollar, but if it happens due to a failure, then it incurs **additional** costs of 4 dollars per replacement. It is given that the lifetime of a lightbulb has a uniform distribution on the interval [0,2].

   Determine the optimal replacement age T (that minimizes the long run mean cost of the replacement) and compute the long run mean replacement cost per unit of time for this choice of T. Compare it to the costs of replacement upon failure.

Solution. Use age replacement strategy from Lecture 20. If the cost of one replacement is K dollars, each replacement due to a failure costs additional c dollars, T is the replacement age and the interrenewal distribution is given by F, then the long run replacement cost (per unit of time) is given by

$$C(T) = \frac{K + cF(T)}{\int_0^T (1 - F(x))dx}. \tag{16}$$

In our particular case, $K = 1$, $c = 4$ and

$$F(t) = \begin{cases} 0, & t \leq 0, \\ t/2, & 0 < t \leq 2, \\ 1, & t > 2, \end{cases} \tag{17}$$

so

$$\int_0^T (1 - F(x))dx = T - \frac{T^2}{4} \tag{18}$$

for $0 \leq T \leq 2$. Therefore,

$$C(T) = \frac{1 + 2T}{T - T^2/4}. \tag{19}$$

Find the minimum

$$C'(T) = \frac{2T - T^2/2 - (1 + 2T)(1 - T/2)}{(T - T^2/4)^2} = \frac{T^2/2 + T/2 - 1}{(T - T^2/4)^2} = 0. \tag{20}$$

Multiplying the numerator by 2, we get that the equation

$$T^2 + T - 2 = 0, \tag{21}$$

which has two solutions, $T = -2$ and $T = 1$. Point $T = 1$ is the point of minimum of $C(T)$ on $(0, 2]$. Therefore, the optimal long run replacement cost per unit of time is equal to

$$C(1) = \frac{1 + 2}{1 - 1/4} = 4. \tag{22}$$

The cost of replacement upon failure is $K + c = 1 + 4 = 5 > 4$.

The failure rate per unit of time is 1 (since the expected length of the interrenewal time is 1). Therefore, the long-run replacement cost per unit of time without using the age replacement policy is $5 \cdot 1 = 5 > 4$.

Therefore, the age replacement policy with the replacement age $T = 1$ will save the company $5 - 4 = 1$ dollars per unit of time in the long run.