MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: General CTMC. Matrix FSA for general MC

Next: PK 6.3, 6.6, Durrett 4.2

Week 3:

- HW2 due Friday, April 21 on Gradescope
- No in-person lecture on Friday, April 21

Matrix exponentials

Results on the previous slide hold for any matrix Q.

Thm. Matrix Q is a Q-matrix

iff $P(t) = e^{tQ}$ is a stochastic matrix $\forall t$

$$P_{ij}(t) \geq 0, \qquad \sum_j P_{ij}(t) = 1 \quad \forall i \text{ and } \forall t \geq 0$$

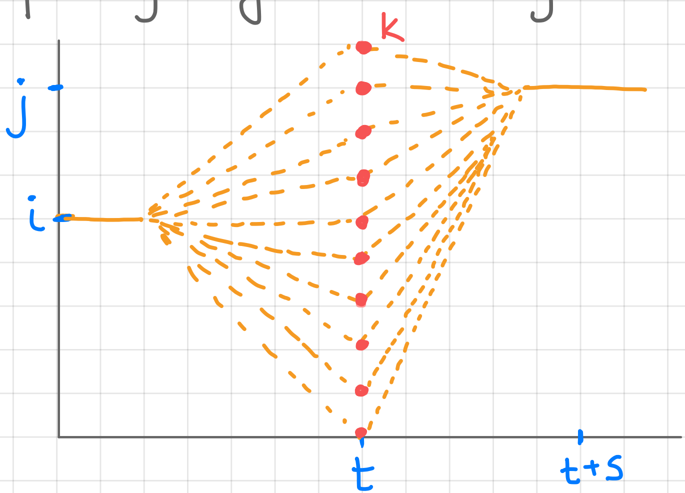
Remarks The semigroup property gives entrywise

$$P_{ij}(t+s) = [P(t)P(s)]_{ij}$$

$$\sum_{k=0}^{N} P_{ik}(t) P_{kj}(s)$$

(if you think about MC

Chapman-Kolmogorov)



Main theorem

Let $P(t)$ be a matrix-valued function $t \geq 0$.

Consider the following properties

(a) $P_{ij}(t) \geq 0$, $\sum_j P_{ij}(t) = 1$ for all $i, j, t \geq 0$

(b) $P(0) = I$

(c) $P(t+s) = P(t)P(s)$ for all $t, s \geq 0$

(d) $\lim_{t \downarrow 0} P(t) = I$ (continuous at 0)

Theorem A. $P(t)$ satisfies (a)–(d)

if and only if

Main theorem. Remarks

This theorem establishes a one-to-one correspondance between matrices $P(t)$ satisfying (a)-(d) and the Q-matrices of the same dimension.

Remarks

1. Conditions (a)-(d) imply that $P(t)$ is differentiable

2. If $P(t) = e^{tQ}$, then $P(h) =$ as $h \to 0$

$$P(h) =$$

Q-matrices and Markov chains

Let $(X_t)_{t\geq 0}$ be a continuous time MC, $X_t \in \{0,1,\dots,N\}$ with right-continuous sample paths

Denote $P_{ij}(t) = P(X_t = j \mid X_0 = i)$, $i,j \in \{0,1,\dots,N\}$

stationary

Then

- $P_{ij}(t)$, $\sum_{j=0}^{N} P_{ij}(t)$ $\left(= \sum_{j=0}^{N} P(X_t = j \mid X_0 = i) \right)$
- $P_{ij}(0) =$ $\left(P(X_0 = j \mid X_0 = i) = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases} \right)$
- $P_{ij}(t+s) = P(X_{t+s} = j \mid X_0 = i)$
- $\lim_{h \downarrow 0} P(X_h = j \mid X_0 = i) =$

Q-matrices and Markov chains (cont.)

$P(t)$ satisfies properties (a)-(d) from Theorem A.

$\Longrightarrow$ there is a Q-matrix Q such that

$$P(t) =$$

In particular,

$$P(h) =$$

This implies the one-to-one correspondance between Q-matrices and continuous time MC with right-continuous sample paths.

Q is called the infinitesimal generator of $(X_t)_{t\geq 0}$

Infinitesimal description of cont. time MC

Let $Q = (q_{ij})_{i,j=0}^{N}$ be a Q-matrix, let $(X_t)_{t\geq 0}$ be right-continuous stochastic process, $X_t \in \{0,1,\dots,N\}$.

We call $(X_t)_{t\geq 0}$ a Markov chain with generator Q, if

(i) $(X_t)_{t\geq 0}$ satisfies the Markov property

(ii) $P(X_{t+h} = j \mid X_t = i) =$

Example

Pure death process

- $P_{i,i-1}(h) = \mu_i h + o(h)$
- $P_{ii}(h) = 1 - \mu_i h + o(h)$
- $P_{ij}(h) = o(h)$ for $j \notin \{i-1, i\}$

The corresponding Q-matrix

$$Q = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

Sojourn time description

Let $Q = (q_{ij})_{i,j=0}^{N}$ be a Q-matrix. Denote $q_i = \sum_{j \neq i} q_{ij}$

so that

$$Q = \begin{pmatrix} & q_{01} & q_{02} & \cdots \\ q_{10} & & q_{12} & \cdots \\ q_{20} & q_{21} & & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

$q_0 = \sum_{i \neq 0} q_{0i}$

$\vdots$

Denote $Y_k := X_{w_k}$ (jump chain).

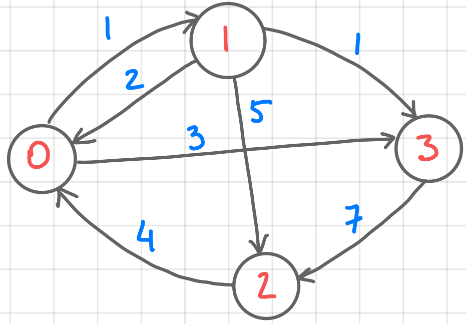
Then the MC with generator matrix Q has the following equivalent jump and hold description

- sojourn times S_k are independent r.v.
  with $P(S_k > t \mid Y_k = i) =$

- transition probabilities $P(Y_{k+1} = j \mid Y_k = i) =$

Example

$$\begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ \hline 0 & -4 & 1 & 0 & 3 \\ 1 & 2 & -8 & 5 & 1 \\ 2 & 4 & 0 & -4 & 0 \\ 3 & 0 & 0 & 7 & -7 \end{array}$$



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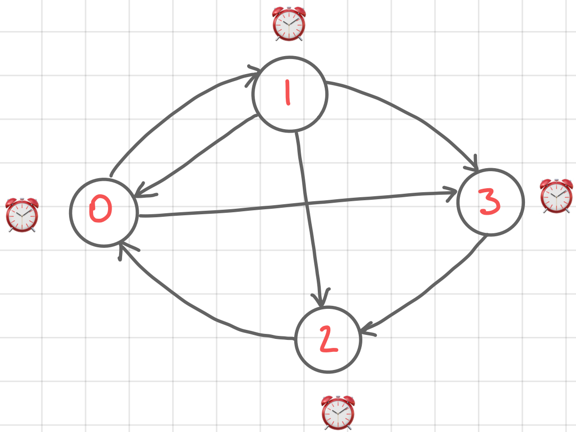
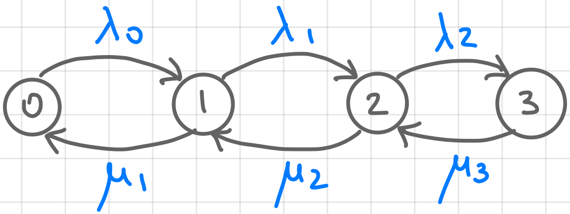
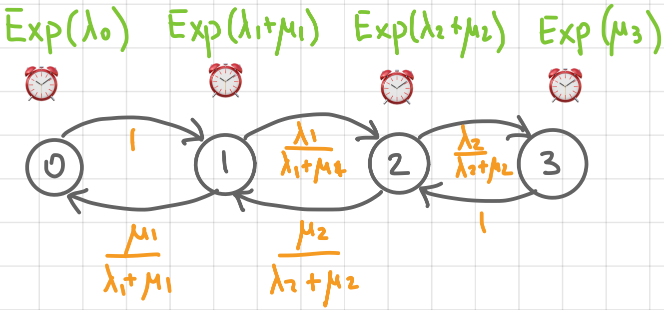


Example

Birth and death process on $\{0,1,2,3\}$

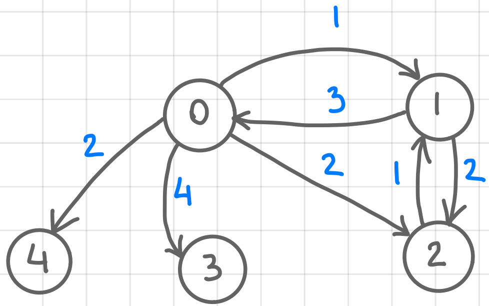


$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 \\ & & \mu_3 & -\mu_3 \end{pmatrix}$$



General continuous time finite state MCs

Rate diagram



Generator

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} -9 & 1 & 2 & 4 & 2 \\ 3 & -5 & 2 & & \\ & 1 & -1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \end{array}$$

Infinitesimal description

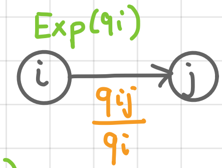
$$P_{ij}(h) = q_{ij}h + o(h) \quad , i \neq j$$

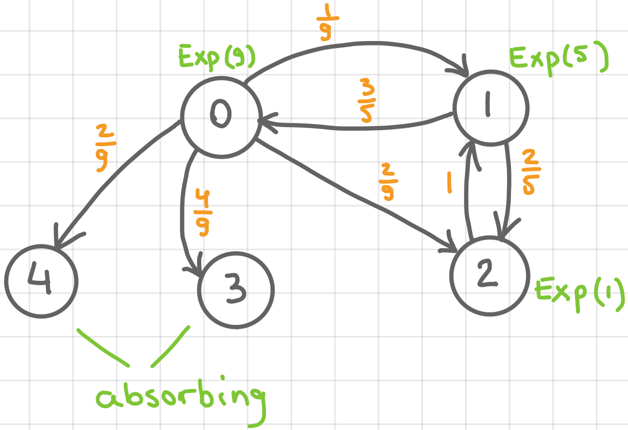
$$P_{ii}(h) = 1 - q_i h + o(h)$$

$$P_{02}(h) = 2h + o(h)$$

$$P_{00}(h) = 1 - 9h + o(h)$$

Jump and hold





Absorption probabilities for finite state chains

By considering the jump chain $(Y_n)_{n\geq 0}$ with $Y_n = X_{W_n}$ and its transition probabilities $P(Y_{n+1}=j \mid Y_n = i) = \frac{q_{ij}}{q_i}$ we can apply the first step analysis to compute, e.g., the absorption probabilities (similarly as for B&D)

If state i is absorbing, then $q_{ij}=0$ for all $j\neq i$ (no jumps from state i), so $q_i = q_{ii} = 0$. Let Q be given by

$$Q = \begin{array}{cc} & \begin{array}{ccc|ccc} 0 & \cdots & K-1 & K & \cdots & N \end{array} \\ \begin{array}{c} 0 \\ \vdots \\ K-1 \\ K \\ \vdots \\ N \end{array} & \left(\begin{array}{ccc|ccc} -q_0 & & & \cdots & q_{ij} & \\ \vdots & & & & \vdots & \\ q_{ij} & \cdots & -q_{K-1} & & & \\ \hline & & & 0 & & 0 \\ & 0 & & & \ddots & \\ & & & & & 0 \end{array}\right) \end{array}$$

with $\{0,\dots,K-1\}$ transient, $\{K,\dots,N\}$ absorbing

Absorption probabilities for finite state chains

$$Q=\begin{array}{c|ccc|ccc}
 & 0 & \cdots & K-1 & K & \cdots & N\\
\hline
0 & -q_0 & & & & \cdots & q_{ij}\\
\vdots & \vdots & & & & & \vdots\\
K-1 & q_{ij} & \cdots & -q_{K-1} & & & \\
\hline
K & & & & 0 & & 0\\
\vdots & & 0 & & & \ddots & \\
N & & & & & & 0
\end{array}$$

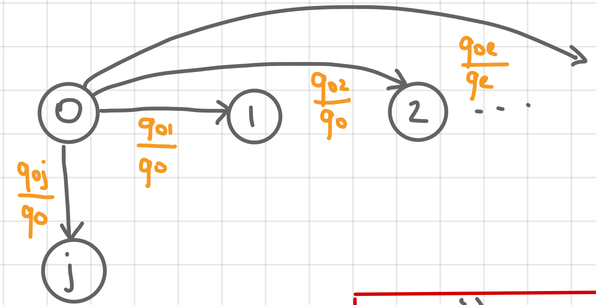
Let $i \in \{0,\dots,K-1\}$, $j \in \{K,\dots,N\}$.

Let $M = \min\{n : Y_n \in \{K,\dots,N\}\}$

Denote $u_i^{(j)} = P(Y_M = j \mid X_0 = i)$.

Then FSA leads to the system

Jump chain



$$u_i^{(j)} = P(Y_M = j \mid Y_0 = i)$$

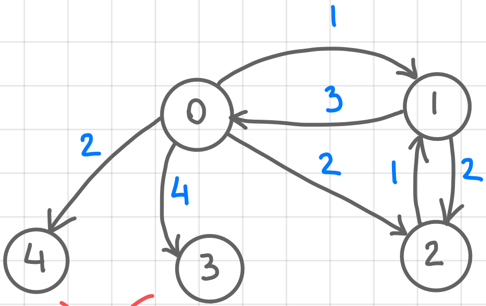
$$=$$

$$\boxed{u_i^{(j)} = \frac{q_{ij}}{q_i} + \sum_{\substack{l=0\\ l\neq i}}^{K-1} \frac{q_{il}}{q_i}\, u_l^{(j)}}$$

$P(Y_{n+1}=j \mid Y_n=i)$ $\qquad$ $P(Y_{n+1}=l \mid Y_n=i)$

Example

Rate diagram



absorbing

Generator

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} -9 & 1 & 2 & 4 & 2 \\ 3 & -5 & 2 & & \\ & 1 & -1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \end{array}$$

Compute $P(Y_M = 3)$ if $P(X_0 = i) = p_i$ for $i = 0,1,2$

$\sum p_i = 1$

Denote $u_i = P(Y_M = 3 \mid Y_0 = i)$.

$$\begin{cases} u_0 = \\ u_1 = \\ u_2 = \end{cases}$$

$$\begin{cases} \\ u_2 = u_1 , \end{cases}$$

$P(Y_M = 3) =$

Mean time to absorption

Similar analysis as was applied to B&D processes can be used to compute the mean time to absorption: before each jump from step i to state j the process sojourns on average in state i.

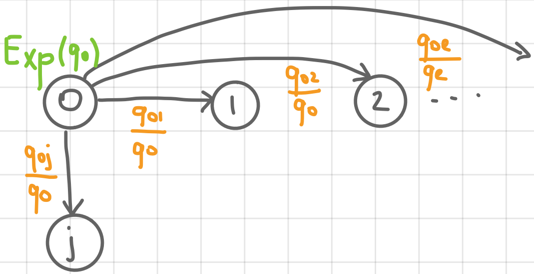
$$Q = \begin{array}{c} \\ 0 \\ \vdots \\ K-1 \\ K \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & \cdots & K-1 & K & \cdots & N \end{array} \\ \left(\begin{array}{ccc|ccc} -q_0 & & & \cdots & q_{ij} & \\ \vdots & & & & \vdots & \\ q_{ij} & \cdots & -q_{K-1} & & & \\ \hline & & & 0 & & 0 \\ & 0 & & & \ddots & \\ & & & & & 0 \end{array}\right) \end{array}$$

Let $T = \min\{t : X_t \in \{K, \dots, N\}\}$

$M = \min\{n : Y_n \in \{K, \dots, N\}\}$

Denote $w_i =$

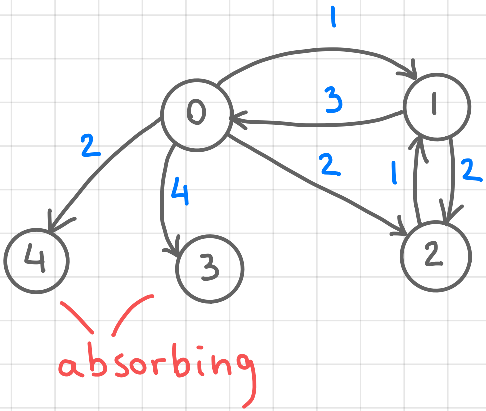
Then FSA gives



$$\boxed{w_i = \qquad\qquad\qquad\qquad}$$

Example

Rate diagram



Generator

$$Q = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & -9 & 1 & 2 & 4 & 2 \\ 1 & 3 & -5 & 2 & & \\ 2 & & 1 & -1 & & \\ 3 & & & & 0 & \\ 4 & & & & & 0 \end{array}$$

$T = \min\{t : X_t \in \{3,4\}\}$

$w_i = E(T \mid X_0 = i)$

$$\begin{cases} w_0 = \\ w_1 = \\ w_2 = \end{cases}$$

$$\begin{cases} \\ \\ w_2 = 1 + w_1 \end{cases}$$