MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Birth processes. Yule process

Next: PK 6.2-6.3

Week 1:

- join Piazza

Description of the birth processes via sojourn times

Theorem

Let $(\lambda_k)_{k\geq 0}$ be a sequence of positive numbers. Let $(X_t)_{t\geq 0}$ be a non-decreasing right-continuous process, $X_0 = 0$, taking values in $\{0,1,2\ldots\}$. Let $(S_i)_{i\geq 0}$ be the sojourn times associated with $(X_t)_{t\geq 0}$, and define $W_\ell = \sum_{i=0}^{\ell-1} S_i$.

Then conditions

(a) $S_0, S_1, S_2, \ldots$ are independent exponential r.v.s of rate $\lambda_0, \lambda_1, \ldots$, $S_k \sim \mathrm{Exp}(\lambda_k)$

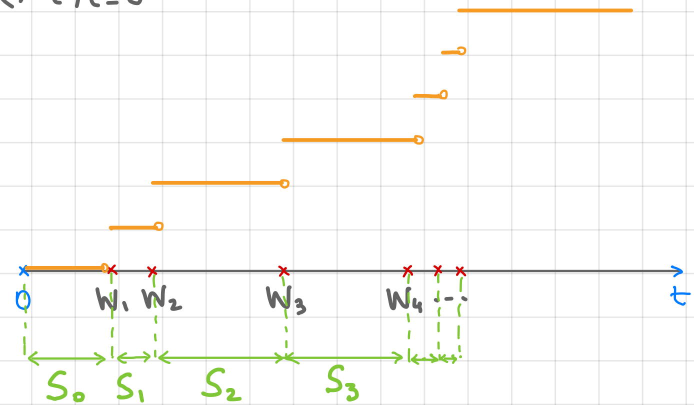
(b) $X_{W_i} = i$

are equivalent to

(c) $(X_t)_{t\geq 0}$ is a pure birth process with parameters (λ_k)

Explosion

$(X_t)_{t\geq 0}$



population becomes infinite in finite time

Thm. Let $(X_t)_{t\geq 0}$ be a pure birth process of rates $(\lambda_k)_{k\geq 0}$.

Then

Birth processes and related differential equations

$P_n(t)$ satisfies the following system of differential eqs. with initial conditions

$$(*)\quad \begin{cases} P_0'(t) = -\lambda_0 P_0(t) & P_0(0) = 1 \\ P_1'(t) = -\lambda_1 P_1(t) + \lambda_0 P_0(t) & P_1(0) = 0 \\ P_2'(t) = -\lambda_2 P_2(t) + \lambda_1 P_1(t) & P_2(0) = 0 \\ \vdots & \vdots \\ P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) & P_n(0) = 0 \\ \vdots & \vdots \end{cases}$$

Solving this system gives the p.m.f. of X_t for any t

Solving the system of differential equations (*)

$$(*)\quad \begin{cases} P_0'(t) = -\lambda_0 P_0(t), & P_0(0) = 1 \\ P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t), & P_n(0) = 0 \quad \text{for } n \geq 1 \end{cases}$$

$P_0(t)$:

$$P_0'(t) =$$

$$\frac{P_0'(t)}{P_0(t)} =$$

$$g'(t) =$$

$$g(t) =$$

Solving the system of differential equations (*)

$P_n(t)$, $n \geq 1$

Consider the function $Q_n(t) =$

$(Q_n(t))' =$

$Q_n(t) =$

$\hookrightarrow P_n(t) =$ $\leftarrow$ apply recursively

$$P_1(t) = e^{-\lambda_1 t} \int_0^t \lambda_0 e^{\lambda_1 s} e^{-\lambda_0 s} ds = e^{-\lambda_1 t} \int_0^t \lambda_0 e^{(\lambda_1 - \lambda_0)s} ds$$ (if $\lambda_1 \neq \lambda_0$)

$$= e^{-\lambda_1 t} \frac{\lambda_0}{\lambda_1 - \lambda_0} \left(e^{(\lambda_1 - \lambda_0)t} - 1 \right) = \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} - \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_1 t}$$

General solution to $(*)$

Assume that $\lambda_i \neq \lambda_j$ for $i \neq j$.

Then for $n \geq 1$

$$P_n(t) = \lambda_0 \cdots \lambda_{n-1} \left(B_{0n} e^{-\lambda_0 t} + \cdots + B_{nn} e^{-\lambda_n t} \right)$$

$$B_{kn} =$$

$P_1(t) =$

$P_2(t) =$

$\vdots$

The Yule process

<u>Setting</u>: In a certain population each individual during any (small) time interval of length h gives a birth to one new individual with probability $\beta h + o(h)$, independently of other members of the population. All members of the population live forever. At time 0 the population consists of one individual.

<u>Question</u>: What is the distribution on the size of the population at a given time t?

The Yule process

Let X_t, $t \geq 0$, be the size of the population at time t.

$X_0 = 1$ (start from one common ancestor).

Compute $\tilde{P}_n(t) = P(X_t = n \mid X_0 = 1)$

If $X_t = n$, then during a time interval of length h

(a) $P(X_{t+h} = n+1 \mid X_t = n) =$

(b) $P(X_{t+h} = n \mid X_t = n) =$

(c) $P(X_{t+h} > n+1 \mid X_t = n) =$

all n indiv. give 0 births

(b) $P(0 \text{ births} \mid X_t = n) = (1 - \beta h + o(h))^n = 1 - n\beta h + o(h)$

(a), (b), (c) $\Rightarrow (X_t)_{t \geq 0}$ is a pure birth process with rates

$P_n(t)$ satisfies the system of differential equations

The Yule process

$$(*)\begin{cases} \tilde{P}_1'(t) = & P_1(0) = \\ \tilde{P}_2'(t) = & P_2(0) = \\ \vdots & \vdots \\ \tilde{P}_n'(t) = & P_n(0) = \\ \vdots & \vdots \\ . & \end{cases}$$

The same system with shifted indices

$\tilde{P}_1(t) =$ $\qquad \tilde{P}_n(t) =$ $\qquad$ with $\lambda_n =$

$$P_n(t) = \lambda_0 \cdots \lambda_{n-1} \left(B_{0n} e^{-\lambda_0 t} + \cdots + B_{nn} e^{-\lambda_n t} \right) \qquad \lambda_0 \cdots \lambda_{n-1} =$$

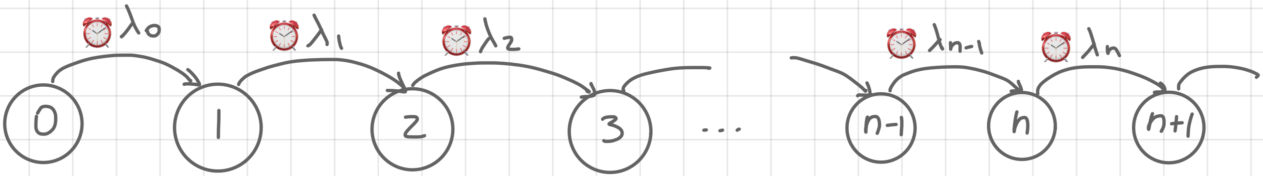
$$B_{kn} = \prod_{\substack{\ell=0 \\ \ell \neq k}}^{n} \frac{1}{\lambda_\ell - \lambda_k} \qquad B_{kn} =$$

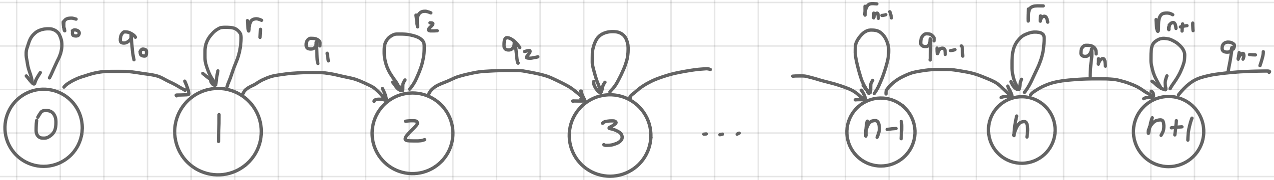
The Yule process

$$P_n(t) = \lambda_0 \cdots \lambda_{n-1} \left(B_{0n} e^{-\lambda_0 t} + \cdots + B_{nn} e^{-\lambda_n t} \right)$$

$$= \sum_{k=0}^{n} \cancel{\beta^n}\, n! \frac{(-1)^k}{\cancel{\beta^n}\, k!\,(n-k)!} e^{-\beta(k+1)t}$$

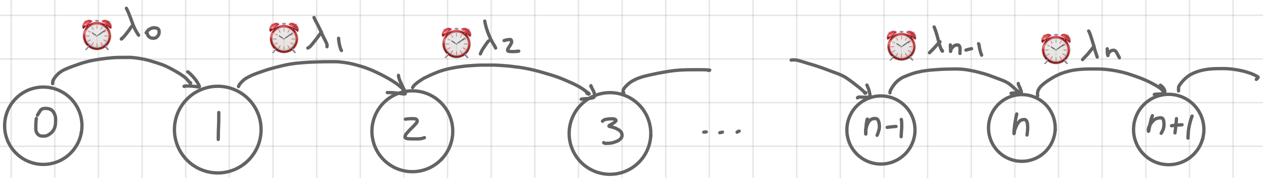
Graphical representation. Exponential sojourn times





Pure death processes

Pure birth process



What if the chain moves in the opposite direction?

Pure death process:

- exponential sojourn times with rates μ_i
- only negative jumps of magnitude 1 allowed

Pure death processes

Infinitesimal description:

Pure death process $(X_t)_{t\geq 0}$ of rates $(\mu_k)_{k=1}^{N}$ is a continuous time MC taking values in $\{0, 1, 2, \ldots, N-1, N\}$ (state 0 is absorbing) with stationary infinitesimal transition probability functions

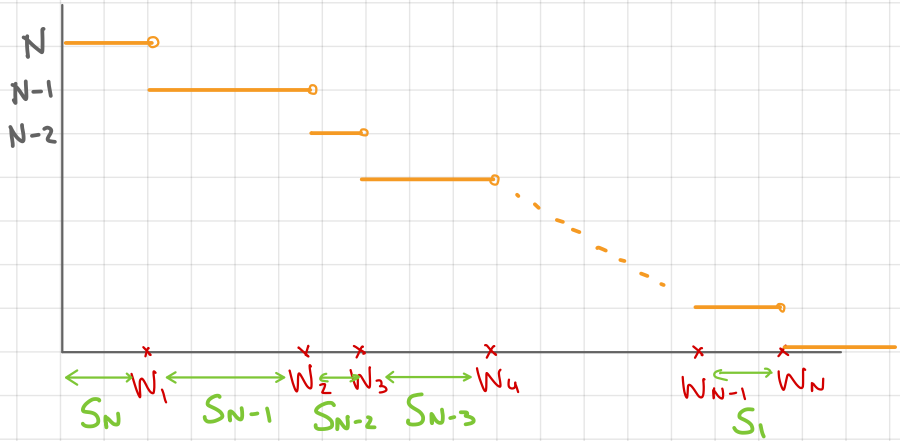
(a) $P_{k,k-1}(h) =$ $\qquad k = 1, \ldots, N$

(b) $P_{kk}(h) =$ $\qquad , \; k = 1, \ldots, N$

(c) $P_{kj}(h) =$ $\qquad$ for $j > k$.

State 0 is absorbing $(\mu_0 = 0)$

Pure death process



$S_k \sim \text{Exp}(\mu_k)$

Sojourn time / jump description:

Pure death process of rates $(\mu_k)_{k=1}^N$ is a nonincreasing right-continuous process taking values in $\{0, 1, \ldots, N\}$

- with sojourn times $S_1, S_2, S_3, \ldots, S_N$ being independent exponential r.v.s of rates $\mu_1, \mu_2, \ldots, \mu_N$ and
- jumps $X_{W_{i+1}} - X_{W_i} = -1$ of magnitude 1

Differential equations for pure birth processes

Define $P_n(t) = P(X_t = n \mid X_0 = N)$ distribution of X_t

starting in state N

(a), (b), (c) implies (check)

$$\begin{cases} P_n'(t) = & \text{for } n = 0 \dots N-1 \quad (\text{note that } \mu_0 = 0) \\ P_N'(t) = & \end{cases}$$

Initial conditions :

Solve recursively: $P_N(t) = \qquad \rightarrow P_{N-1}(t) \rightarrow \cdots \rightarrow P_0(t)$

General solution (assume $\mu_i \neq \mu_j$)

$$P_n(t) = \mu_{n+1} \cdots \mu_N \left(A_{nn} e^{-\mu_n t} + \cdots + A_{N,n} e^{-\mu_N t} \right), \quad A_{kn} = \prod_{\substack{\ell = n \\ \ell \neq k}}^{N} \frac{1}{\mu_\ell - \mu_k}$$

Linear death process

Similar to Yule process:

death rate is proportional to the size of the population

Compute $P_n(t)$: • $\mu_{n+1} \cdots \mu_N = \alpha^{N-n} \frac{N!}{n!}$

• $A_{kn} = \prod_{\substack{\ell=n \\ \ell \neq k}}^{N} \frac{1}{\mu_\ell - \mu_k} = \frac{1}{\alpha^{N-n} (-1)^{n-k} (k-n)!\,(N-k)!}$ $\quad \{ \mu_\ell - \mu_k = \alpha(\ell - k)$

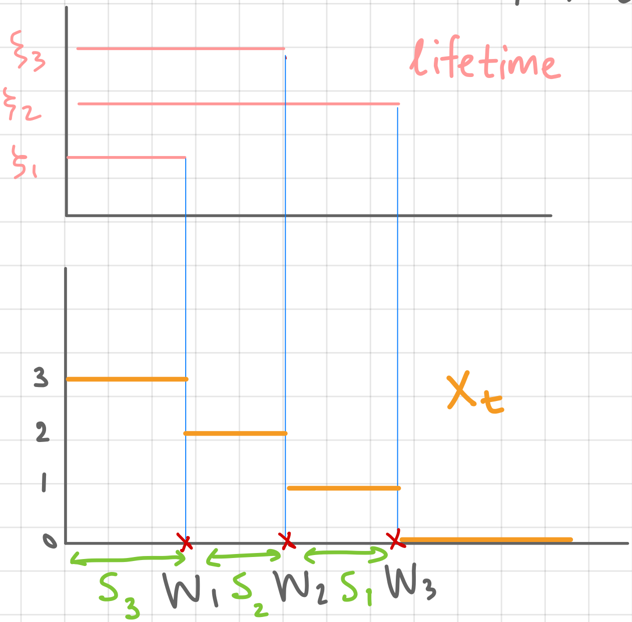
• $P_n(t) = \alpha^{N-n} \frac{N!}{n!} \cdot \frac{1}{\alpha^{N-n}} \sum_{k=n}^{N} \frac{1}{(-1)^{n-k} (k-n)!\,(N-k)!} \cdot e^{-k\alpha t}$ $\quad \left\{ \begin{array}{l} j = k-n \\ k = j+n \end{array} \right.$

$$= \frac{N!}{n!} \sum_{j=0}^{N-n} \frac{(-1)^j e^{-(j+n)\alpha t}}{j!\,(N-n-j)!}$$

$$= \frac{N!}{n!} e^{-n\alpha t} \sum_{j=0}^{N-n} \frac{1}{j!\,(N-n-j)!} \left(-e^{-\alpha t}\right)^j = \frac{N!}{n!\,(N-n)!} e^{-n\alpha t} \left(1 - e^{-\alpha t}\right)^{N-n}$$

Interpretation of $X_t \sim Bin(n, e^{-\alpha t})$

Consider the following process: Let ξ_i, $i = 1 \dots N$, be i.i.d. r.v.s, $\xi_i \sim Exp(\alpha)$. Denote by X_t the number of ξ_i's that are bigger than t (ξ_i is the lifetime of an individual, X_t = size of the population at t). $X_0 = N$.



Then: $S_k \sim$, independent

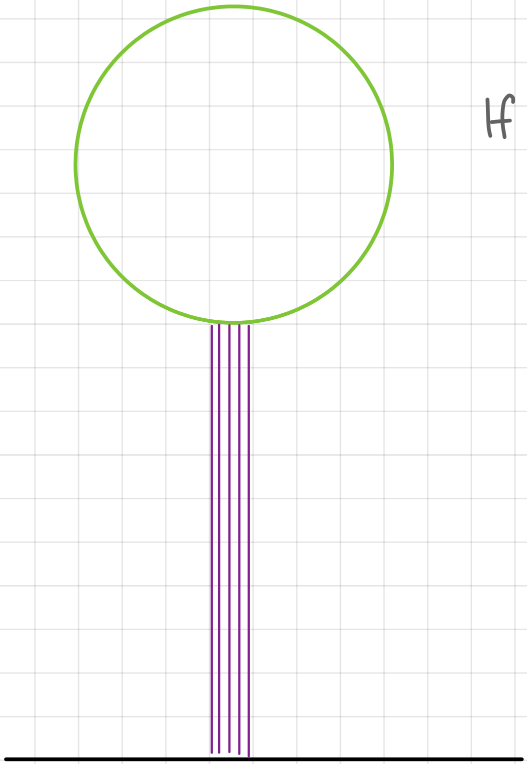
↳ $(X_t)_{t\geq 0}$ is a pure death process

Probability that an individual survives to time t is

Probability that exactly n individuals survive to time t is

$$\binom{N}{n} e^{-\alpha t n}(1-e^{-\alpha t})^{N-n} = P(X_t = n)$$

Example. Cable

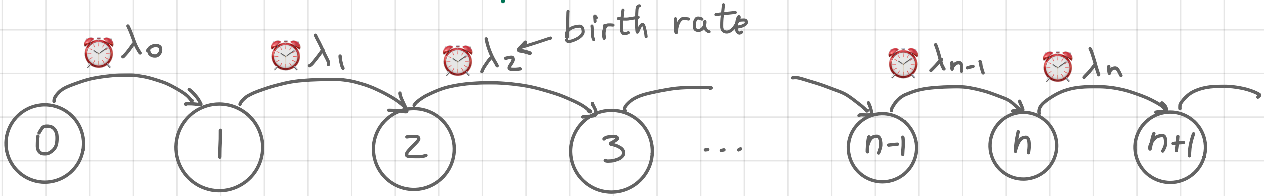


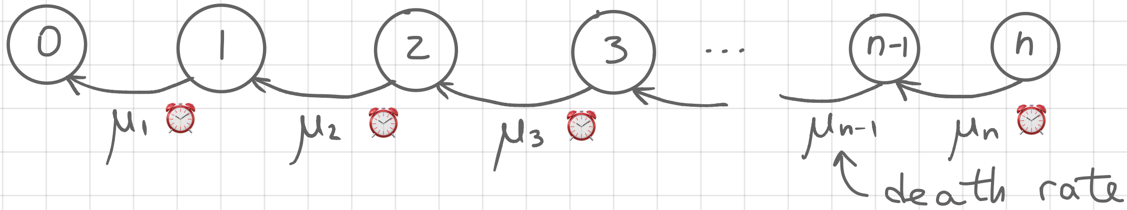
X_t = number of fibers in the cable

If a fiber fails, then this increases the load on the remaining fibers, which results in a shorter lifetime.

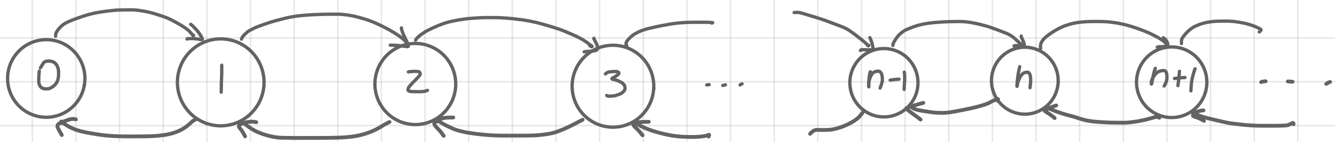
↳ pure death process

Birth and death processes





Combine both



Birth and death processes

Infinitesimal definition

Def. Let $(X_t)_{t\geq 0}$ be a continuous time MC, $X_t \in \{0,1,2,\dots\}$ with stationary transition probabilities. Then $(X_t)_{t\geq 0}$ is called a birth and death process with birth rates (λ_k) and death rates (μ_k) if

1. $P_{i,i+1}(h) =$
2. $P_{i,i-1}(h) =$
3. $P_{i,i}(h) =$
4. $P_{ij}(0) =$ $\quad \left(P(X_0=j \mid X_0=i) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \right)$
5. $\mu_0 = 0$, $\lambda_0 > 0$, $\lambda_i, \mu_i > 0$

Example: Linear growth with immigration

Dynamics of a certain population is described by the following principles:

during any small period of time of length h

- each individual gives birth to one new member with probability independently of other members;
- each individual dies with probability independently of other members;
- one external member joins the population with probability

Can be modeled as a Markov process

Example: Linear growth with immigration

Let $(X_t)_{t\geq 0}$ denote the size of the population.

Using a similar argument as for the Yule/pure death models:

- $P_{n,n+1}(h) =$

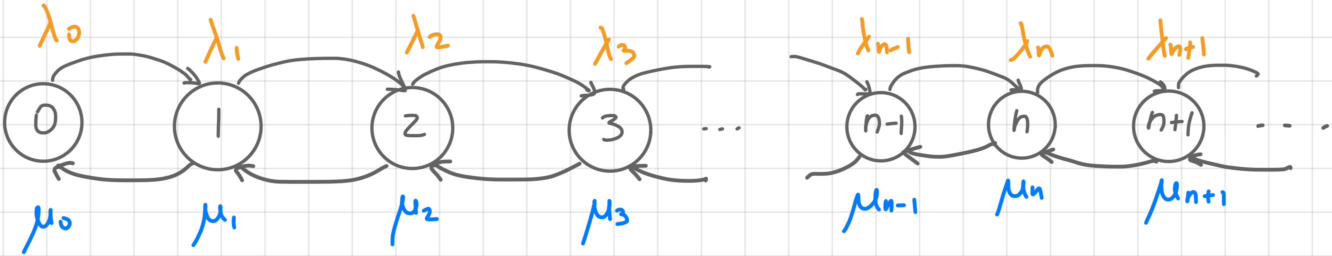
- $P_{n,n-1}(h) =$
- $P_{n,n}(h) =$

↳ birth and death process with

$\lambda_n =$

$\mu_n =$

Alternative (jump and hold) characterization



Sojourn times S_k are independent,

Each transition has two parts

- wait in state i for time $\sim$
- then choose where to go:

go $\rightarrow$ $i+1$ with probability $\frac{\quad}{\quad + \quad}$

go $i-1$ $\leftarrow$ with pobability $\frac{\quad}{\quad + \quad}$