MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

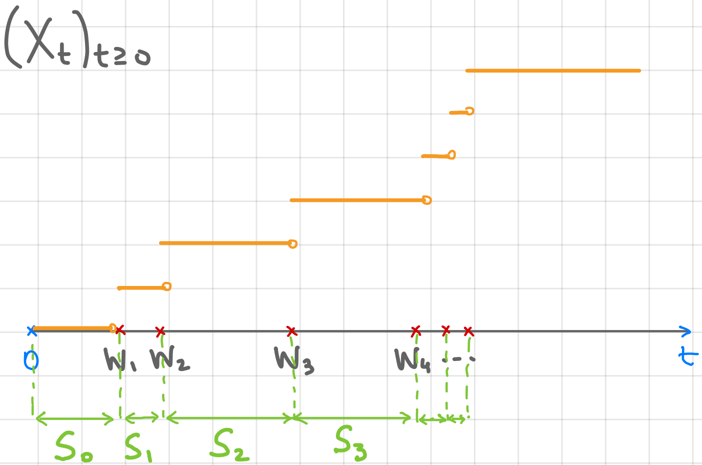
Today: Birth processes. Yule process

Next: PK 6.2-6.3

Week 1:

- join Piazza
- HW1 due Friday, April 14 on Gradescope

Description of the birth processes via sojourn times



W_i – i-th "birth time" S_i – "time between (i-1)-th birth and i-th birth"

$W_i = \sum_{l=0}^{i-1} S_l$ ↳ sojourn times

Alternative way of characterizing $(X_t)_{t\geq 0}$:

- describe the distribution of $(S_i)_{i\geq 0}$
- desribe the jumps $X_{W_{i+1}} - X_{W_i}$

Description of the birth processes via sojourn times

Theorem

Let $(\lambda_k)_{k\geq 0}$ be a sequence of positive numbers. Let $(X_t)_{t\geq 0}$ be a non-decreasing right-continuous process, $X_0 = 0$, taking values in $\{0,1,2\ldots\}$. Let $(S_i)_{i\geq 0}$ be the sojourn times associated with $(X_t)_{t\geq 0}$, and define $W_\ell = \sum_{i=0}^{\ell-1} S_i$.

Then conditions

(a) $S_0, S_1, S_2, \ldots$ are independent exponential r.v.s of rate $\lambda_0, \lambda_1, \ldots$, $S_k \sim \text{Exp}(\lambda_k)$

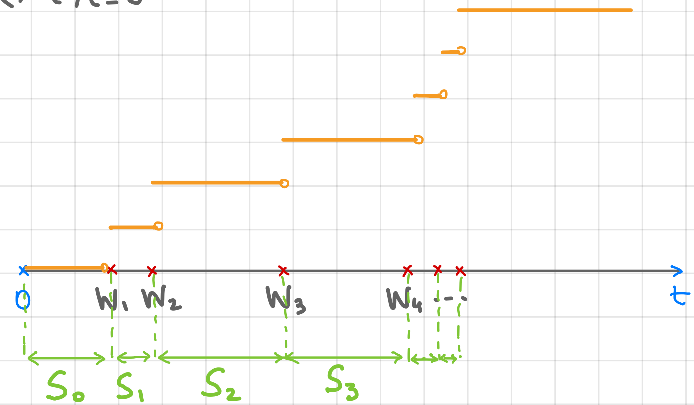
(b) $X_{W_i} = i$

are equivalent to

(c) $(X_t)_{t\geq 0}$ is a pure birth process with parameters (λ_k)

Explosion

$(X_t)_{t\geq 0}$



population becomes infinite in finite time

Thm. Let $(X_t)_{t\geq 0}$ be a pure birth process of rates $(\lambda_k)_{k\geq 0}$.

Then
- if $\sum_{k=0}^{\infty} \frac{1}{\lambda_k} < \infty$, then $P\big((X_t)_{t\geq 0} \text{ explodes}\big) = 1$
- if $\sum_{k=0}^{\infty} \frac{1}{\lambda_k} = \infty$, then $P\big((X_t)_{t\geq 0} \text{ does not explode}\big) = 1$

Hint: $E\left[\sum_{k=0}^{\infty} S_k\right] = \sum_{k=0}^{\infty} \frac{1}{\lambda_k}$

Birth processes and related differential equations

$P_n(t)$ satisfies the following system of differentian eqs. with initial conditions

$$(*)\quad \begin{cases} P_0'(t) = -\lambda_0 P_0(t) & P_0(0) = 1 \\ P_1'(t) = -\lambda_1 P_1(t) + \lambda_0 P_0(t) & P_1(0) = 0 \\ P_2'(t) = -\lambda_2 P_2(t) + \lambda_1 P_1(t) & P_2(0) = 0 \\ \vdots & \vdots \\ P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) & P_n(0) = 0 \\ \vdots & \vdots \end{cases}$$

Solving this system gives the p.m.f. of X_t for any t

$$P_n(t) = P(X_t = n)$$

Solving the system of differential equations (*)

$$(*)\quad \begin{cases} P_0'(t) = -\lambda_0 P_0(t), & P_0(0) = 1 \\ P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t), & P_n(0) = 0 \quad \text{for } n \geq 1 \end{cases}$$

$P_0(t)$:

$$\Big(\underbrace{\log(P_0(t))}_{g(t)}\Big)' = \frac{P_0'(t)}{P_0(t)}$$

$$P_0'(t) = -\lambda_0 P_0(t)$$

$$\frac{P_0'(t)}{P_0(t)} = -\lambda_0$$

$$g'(t) = -\lambda_0$$

$$g(t) = -\lambda_0 t + K = \log(P_0(t))$$

$$P_0(t) = e^{-\lambda_0 t} \cdot \underset{C}{\underset{\shortparallel}{e^{K}}} = C \cdot e^{-\lambda_0 t}, \quad C > 0$$

$$P_0(0) = C = 1$$

$$\Rightarrow \quad P_0(t) = e^{-\lambda_0 t}$$

Solving the system of differential equations (*)

$P_n(t)\,,\ n \geq 1$

Consider the function $Q_n(t) = e^{\lambda_n t} P_n(t)$

$$\left(Q_n(t)\right)' = \lambda_n e^{\lambda_n t} P_n(t) + e^{\lambda_n t} P_n'(t)$$

$$= \cancel{\lambda_n e^{\lambda_n t} P_n(t)} + e^{\lambda_n t}\left(\cancel{-\lambda_n P_n(t)} + \lambda_{n-1} P_{n-1}(t)\right)$$

$$= \lambda_{n-1} P_{n-1}(t)$$

$$Q_n(t) = \int_0^t \lambda_{n-1} e^{\lambda_n s} P_{n-1}(s)\,ds$$

$\hookrightarrow$ $P_n(t) = e^{-\lambda_n t} \int_0^t \lambda_{n-1} e^{\lambda_n s} P_{n-1}(s)\,ds$ $\leftarrow$ apply recursively

$$P_1(t) = e^{-\lambda_1 t} \int_0^t \lambda_0 e^{\lambda_1 s} e^{-\lambda_0 s}\,ds = e^{-\lambda_1 t} \int_0^t \lambda_0\, e^{(\lambda_1 - \lambda_0)s}\,ds \qquad (\text{if } \lambda_1 \neq \lambda_0)$$

$$= e^{-\lambda_1 t} \frac{\lambda_0}{\lambda_1 - \lambda_0}\left(e^{(\lambda_1 - \lambda_0)t} - 1\right) = \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} - \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_1 t}$$

General solution to (*)

Assume that $\lambda_i \neq \lambda_j$ for $i \neq j$.

Then for $n \geq 1$

$$P_n(t) = \lambda_0 \cdots \lambda_{n-1} \left(B_{0n} e^{-\lambda_0 t} + \cdots + B_{nn} e^{-\lambda_n t} \right)$$

$$B_{kn} = \prod_{\substack{\ell=0 \\ \ell \neq k}}^{n} \frac{1}{\lambda_\ell - \lambda_k}$$

$$P_1(t) = \lambda_0 \left(\frac{1}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} + \frac{1}{\lambda_0 - \lambda_1} e^{-\lambda_1 t} \right)$$

$$P_2(t) =$$

$\vdots$

The Yule process

Setting: In a certain population each individual during any (small) time interval of length h gives a birth to one new individual with probability $\beta h + o(h)$, independently of other members of the population. All members of the population live forever. At time 0 the population consists of one individual.

Question: What is the distribution on the size of the population at a given time t?

The Yule process

Let X_t, $t \geq 0$, be the size of the population at time t.

$X_0 = 1$ (start from one common ancestor).

Compute $\tilde{P}_n(t) = P(X_t = n \mid X_0 = 1)$

If $X_t = n$, then during a time interval of length h

(a) $P(X_{t+h} = n+1 \mid X_t = n) = n\beta h + o(h)$

(b) $P(X_{t+h} = n \mid X_t = n) = 1 - n\beta h + o(h)$

(c) $P(X_{t+h} > n+1 \mid X_t = n) = o(h)$

all n indiv. give 0 births

(b) $P(0 \text{ births} \mid X_t = n) = (1 - \beta h + o(h))^n = 1 - n\beta h + o(h)$

(a), (b), (c) $\Rightarrow (X_t)_{t \geq 0}$ is a pure birth process with rates $\lambda_n = \beta n$

$\tilde{P}_n(t)$ satisfies the system of differential equations

The Yule process

$$(*)\quad \begin{cases} \tilde{P}_1'(t) = -\beta \tilde{P}_1(t) & \tilde{P}_1(0) = 1 \\ \tilde{P}_2'(t) = -2\beta \tilde{P}_2(t) + \beta \tilde{P}_1(t) & \tilde{P}_2(0) = 0 \\ \vdots & \vdots \\ \tilde{P}_n'(t) = -n\beta \tilde{P}_n(t) + (n-1)\beta \tilde{P}_{n-1}(t) & \tilde{P}_n(0) = 0 \\ \vdots & \vdots \end{cases}$$

The same system with shifted indices

$$\tilde{P}_1(t) = P_0(t) \qquad \tilde{P}_n(t) = P_{n-1}(t) \qquad \text{with } \lambda_n = \beta(n+1)$$

$$P_n(t) = \lambda_0 \cdots \lambda_{n-1} \left(B_{0n} e^{-\lambda_0 t} + \cdots + B_{nn} e^{-\lambda_n t} \right) \qquad \lambda_0 \cdots \lambda_{n-1} = \beta^n n!$$

$$B_{kn} = \prod_{\substack{\ell=0 \\ \ell \neq k}}^{n} \frac{1}{\lambda_\ell - \lambda_k} \qquad B_{kn} =$$