MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Brownian motion

Next: PK 8.1- 8.2

Week 9:

- homework 6 (due Friday, June 2)

Brownian motion. Definition

Def. **Brownian motion** with diffusion coefficient σ^2 is a continuous time stochastic process $(B_t)_{t\geq 0}$ satisfying

(i) $B(0)=0$, $B(t)$ is continuous as a function of t

(ii) For all $0\leq s<t<\infty$ $B(t)-B(s)$ is a Gaussian random variable with mean 0 and variance $\sigma^2(t-s)$

(iii) The increments of B are independent: if $0=t_0<t_1<\dots<t_n$ then $\{B(t_i)-B(t_{i-1})\}_{i=1}^{n}$ are independent (Gaussian) r.v.s.

$\sigma^2=1 \leftarrow$ standard BM

BM as a Gaussian process

<u>Def</u>. Stochastic process $(X_t)_{t\geq 0}$ is called a Gaussian process if for any $0 \leq t_1 < t_2 < \dots < t_n$

$(X_{t_1}, \dots, X_{t_n})$ is a Gaussian vector, or equivalently for any $c_1, \dots, c_n \in \mathbb{R}$

is a Gaussian r.v.

Recall that the distribution of a Gaussian vector is uniquelly defined by its mean and covariance matrix.

Similarly, each Gaussian process is uniquely described by

$$\mu(t) = E(X_t) \quad \text{and} \quad \Gamma(s,t) = \mathrm{Cov}(X_s, X_t) \geq 0$$

↑ covariance function

BM as a Gaussian process

Proposition BM is a Gaussian process with

and

Proof. For any $0 \le t_1 < t_2 < \cdots < t_n$, $B_{t_j} - B_{t_{j-1}}$ are indep. Gaussian, thus

$$\sum_{i=1}^{n} c_i B_{t_i} =$$

is also Gaussian.

By definition . Let $s < t$.

Then $\Gamma(s,t) =$

$=$

$=$

$=$

Some properties of BM

Proposition. Let $(B_t)_{t\geq 0}$ be a standard BM. Then

(i) For any $s>0$, the process is a BM independent of $(B_u, 0\leq u\leq s)$.

(ii) The process is a BM

(iii) For any $c>0$, the process is a BM

(iv) The process $(X_t)_{t\geq 0}$ defined by for $t>0$ is a BM.

Proof (i) Define $X_t = B_{t+s} - B_s$. Then

$\Rightarrow$ independent Gaussian increments,

$(X_t)_{t\geq 0}$ has continuous paths $\Rightarrow$

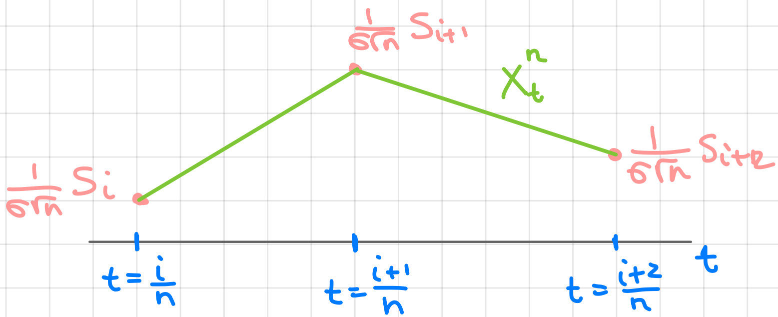
(iv) X_t is Gaussian, for $s<t$

Proof of $\lim_{t\to 0} X_t = 0$ is more technical, thus omitted.

Construction of BM

BM can be constructed as a limit of properly rescaled random walks.

Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. r.v.s, $E(\xi_i)=0$, $Var(\xi_i)=\sigma^2<\infty$. Denote   and define



Theorem (Donsker)

Applying Donsker's theorem

Example Let $(\xi_i)_{i=1}^{\infty}$ be i.i.d. r.v. $P(\xi_i = 1) = P(\xi_i = -1) = 0.5$

$E(\xi_i) = 0$, $Var(\xi_i) = 1$.

Denote $(S_m)_{m \ge 0}$ is a Markov chain.

From the first step analysis of MC we know that for any $-a < 0 < b$

If X^n_t is the process interpolating S_m, then $\forall n$

$P(X^n$ hits $-a$ before $b) =$

$\Rightarrow P(B$ hits $-a$ before $b) =$

$\Rightarrow (\tilde{\xi}_i)_{i=1}^{\infty}$, $E(\tilde{\xi}_i) = 0$, $Var(\tilde{\xi}_i) = 1$, $P(\tilde{S}$ hits $-a$ before $b) \approx \frac{b}{a+b}$

BM as a martingale

Let $(X_t)_{t\geq 0}$ be a continuous time stochastic process. We say that $(X_t)_{t\geq 0}$ is a martingale if $E(|X_t|) < \infty$ $\forall t \geq 0$ and

<u>Proposition</u> Let $(B_t)_{t\geq 0}$ be a standard BM. Then

(i)

(ii)

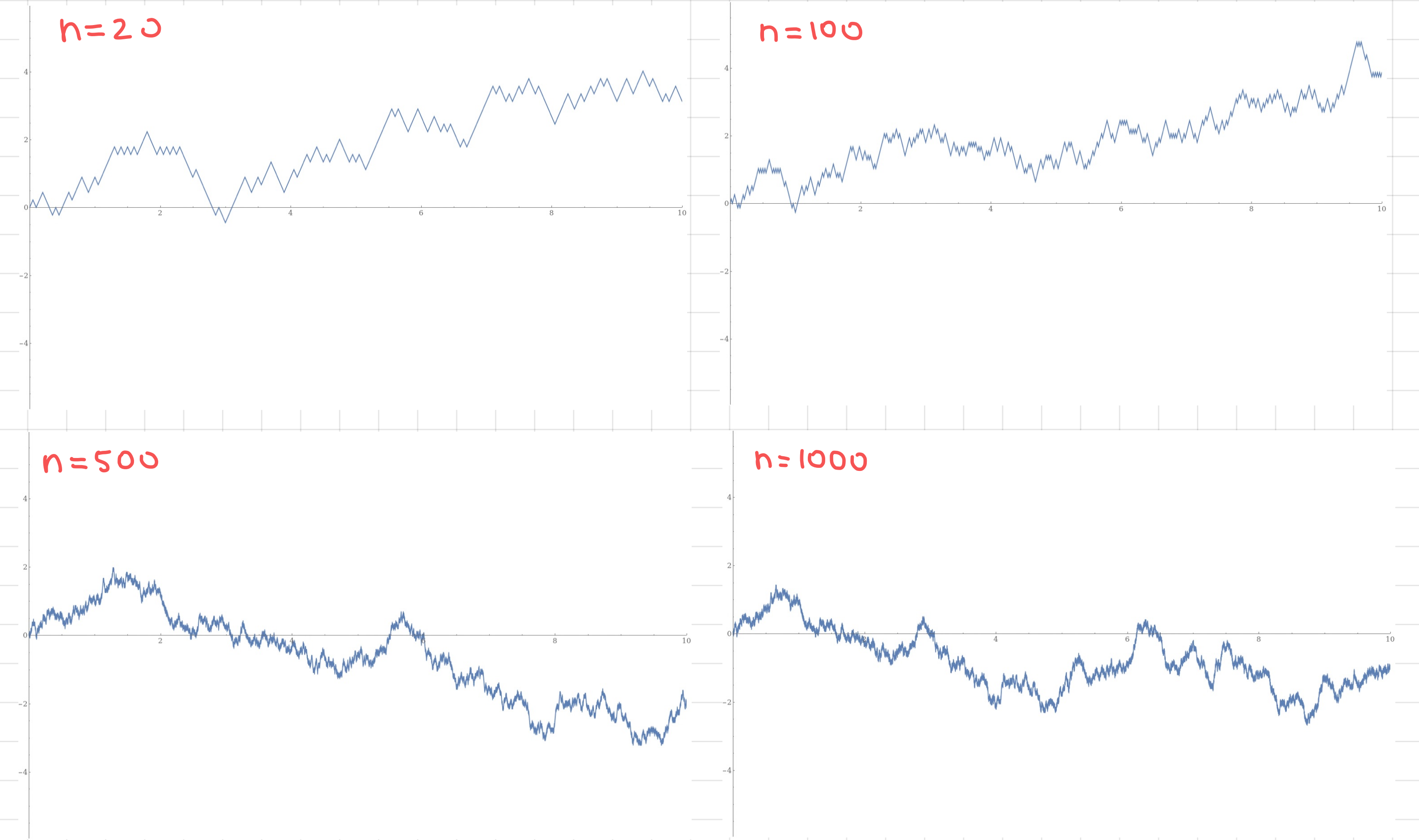
"Proof": $E(B_t \mid \{B_u, 0 \leq u \leq s\}) =$

$E(B_t^2 - t \mid \{B_u, 0 \leq u \leq s\}) =$

$=$

<u>Thm</u> (Lévy) Let $(X_t)_{t\geq 0}$ be a continuous martingale such that $(X_t^2 - t)_{t\geq 0}$ is a martingale.

Approximating a BM with random walks X_t^n



Stopping times and the strong Markov property (lec. 3)

<u>Def</u> (Informal). Let $(X_t)_{t\geq 0}$ be a stochastic process and let $T \geq 0$ be a random variable. We call T a **stopping time** if the event

$$\{T \leq t\}$$

can be determined from the knowledge of the process up to time t (i.e., from $\{X_s : 0 \leq s \leq t\}$)

<u>Examples</u>: Let $(X_t)_{t\geq 0}$ be right-continuous

1. $\min\{t \geq 0 : X_t = x\}$ is a stopping time

2. $\sup\{t \geq 0 : X_t = x\}$ is not a stopping time

Stopping times and the strong Markov property (lec. 3)

Theorem (no proof)

Let $(X_t)_{t\geq 0}$ be a Markov process, let T be a stopping time of $(X_t)_{t\geq 0}$. Then, conditional on $T<\infty$ and $X_T = x$,

$$(X_{T+t})_{t\geq 0}$$

(i) is independent of $\{X_s,\ 0\leq s\leq T\}$

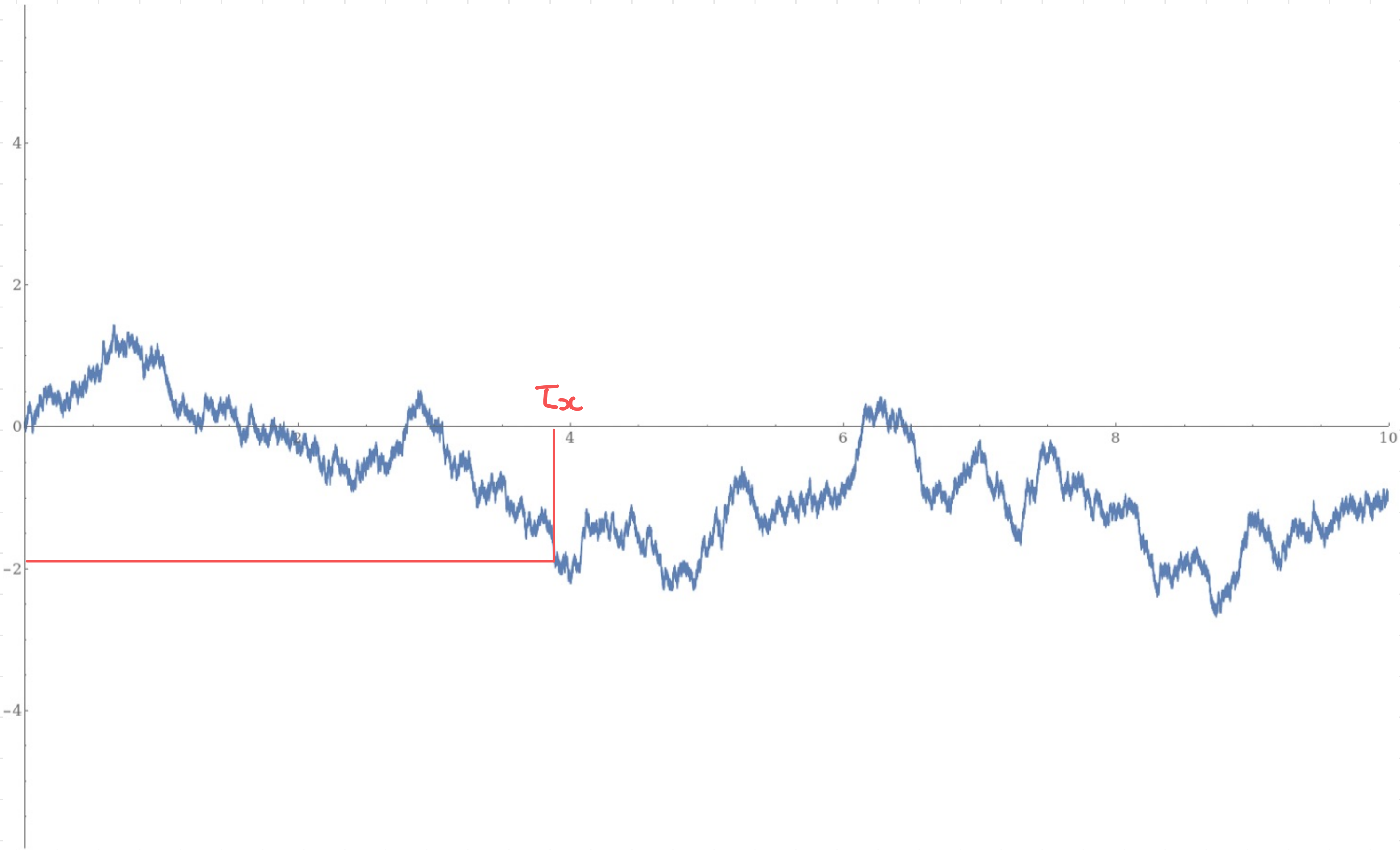
(ii) has the same distribution as $(X_t)_{t\geq 0}$ starting from x

Example $(B_t)_{t\geq 0}$ is Markov. For any $x\in\mathbb{R}$ define

$$\tau_x = \min\{t : B_t = x\}.$$ Then

- $(B_{t+\tau_x} - B_{\tau_x})_{t\geq 0}$ is a BM starting from x
- $(B_{t+\tau_x} - B_{\tau_x})_{t\geq 0}$ is independent of $\{B_s,\ 0\leq s\leq \tau_x\}$

(independent of what B was doing before it hit x)


τ_x
4
2
0
-2
-4
4
6
8
10

Reflection principle

<u>Thm</u>. Let $(B_t)_{t\geq 0}$ be a standard BM. Then for any $t \geq 0$ and $x > 0$

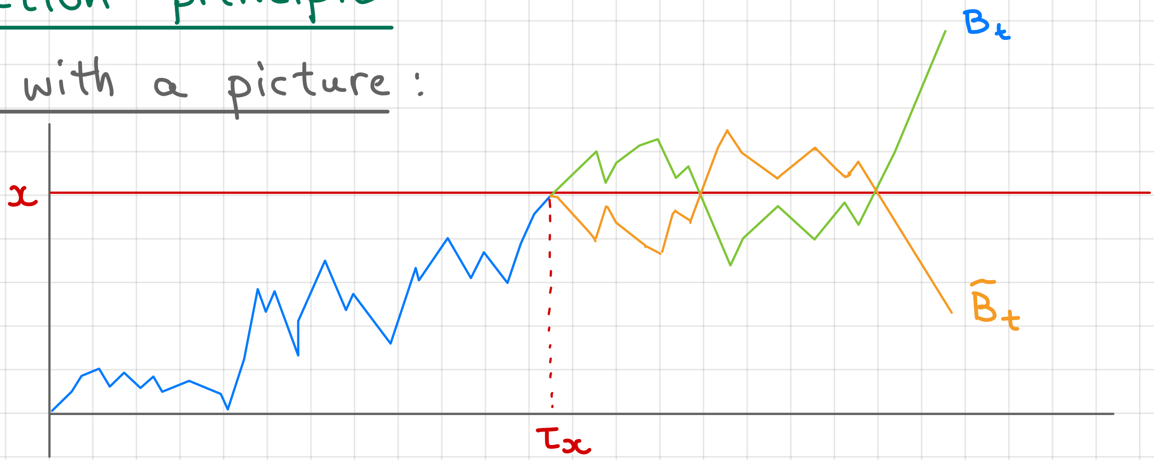
<u>Proof</u>. Let $\tau_x = \min\{t : B_t = x\}$. Note that τ_x is a stopping time and is uniquely determined by $\{B_u, 0 \leq u \leq \tau_x\}$
From the definition of τ_x, . Then

$$P\left(\max_{0 \leq u \leq t} B_u \geq x,\ B_t < x\right) =$$

Now $P\left(\max_{0 \leq u \leq t} B_u \geq x\right) =$

Reflection principle

Proof with a picture:



If $(B_t)_{t\geq 0}$ is a BM, then $(\tilde{B}_t)_{t\geq 0}$ is a BM, where

$$\tilde{B}_t = \begin{cases} B_t, & t \leq \tau_x \\ B_{\tau_x} - (B_t - B_{\tau_x}), & t > \tau_x \end{cases}$$

$\Rightarrow$ to each sample path with $\max\limits_{0\leq u\leq t} B_u > x$ and $B_t > x$ we associate a unique path with $\max\limits_{0\leq u\leq t} \tilde{B}_u > x$ and $\tilde{B}_t < x$, so

$$P\left(\max_{0\leq u\leq t} B_u \geq x,\ B_t < x\right) = P(B_t > x) \Rightarrow P\left(\max_{0\leq u\leq t} B_u \geq x\right) = 2P(B_t \geq x)$$