MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Martingales

Next: PK 8.1

Week 8:

Martingales

Definition. A stochastic process $(X_n, n \geq 0)$ is a martingale if for $n = 0, 1, \ldots$

(a) $E(|X_n|) < \infty$

(b) $E(X_{n+1} \mid X_0, X_1, \ldots, X_n) = X_n$

Thm. Let $(X_n)_{n \geq 0}$ be a martingale with nonnegative values. For any $\lambda > 0$ and $m \in \mathbb{N}$

$$P\left(\max_{0 \leq n \leq m} X_n \geq \lambda\right) \leq \frac{E(X_0)}{\lambda} \qquad (1)$$

and

$$P\left(\max_{n \geq 0} X_n \geq \lambda\right) \leq \frac{E(X_0)}{\lambda} \qquad (2)$$

Example

A gambler begins with a unit amount of money and faces a series of independent fair games. In each game the gamblers bets fraction p of his current fortune, wins with probability $\frac{1}{2}$, loses with probability $\frac{1}{2}$. Estimate the probability that the gambler ever doubles the initial fortune.

Denote by $Z_n, n \geq 0$, the gambler's fortune after n-th game.

Denote

Then

Martingale transform

In the previous example the stake in n-th game is $p Z_{n-1}$. What if we choose another strategy?

Def Let $(X_n)_{n\geq 0}$ be a nonnegative martingale, and let $(C_n)_{n\geq 0}$ be a stochastic process with $C_n = f_n(X_0, \dots, X_{n-1})$. Then the stochastic process

is called the

Think of
- $X_k - X_{k-1}$ as the winning per unit stake in k-th game
- C_k as your stake in k-th game
  decision is made based on the previous history
- $(C \cdot X)_n$ as total winnings up to time n

Martingale transform

Prop. Let $Z_n = X_0 + (C \bullet X)_n$. Let $C_k > 0$ bounded if $Z_{k-1} > 0$ and $C_k = 0$ if $Z_{k-1} = 0$. Then $(Z_n)_{n \geq 0}$ is a martingale

Proof: $E(Z_{n+1} \mid Z_0, \dots, Z_n) =$

$=$

Note that

If $Z_n > 0$, then $C_1 > 0, \dots, C_n > 0$,

and

$E(Z_{n+1} \mid Z_0, \dots, Z_n) =$

$=$

If $Z_n = 0$, then $C_{n+1} = 0$ and $E(Z_{n+1} \mid Z_0, \dots, Z_n) = 0 = Z_n$

Convergence of nonnegative martingales

Thm.

If $(X_n)_{n\geq 0}$ is a nonnegative (super)martingale, then with probability 1

and

Example

An urn initially contains one red ball and one green ball. Choose a ball and return it to the urn together with another ball of the same color. Repeat. Denote by X_n the fraction of red ball after n iterations.

Example (cont.)

(i) $(X_n)_{n\geq 0}$ is a martingale

Denote by R_n the number of red balls after n-th iteration

$$R_n =$$

Then

$$E(X_{n+1} \mid X_0, \dots, X_n) =$$

$$=$$

(ii) X_n is nonnegative $\Rightarrow$

(iii) Compute the distribution of X_∞

Brownian motion. History

- Critical observation: Robert Brown (1827), botanist, movement of pollen grains in water
- First (?) mathematical analysis of Brownian motion: Louis Bachelier (1900), modeling stock market fluctuations
- Brownian motion in physics: Albert Einstein (1905) and Marian Smoluchowski (1906), explained the phenomenon observed by Brown
- First rigorous construction of mathematical Brownian motion: Norbert Wiener (1923)

Brownian motion $\underset{\substack{\uparrow \\ \text{in mathematics}}}{=}$ Wiener process

Brownian motion. Motivation

- almost all interesting classes of stochastic processes contain Brownian motion: BM is a
  - martingale
  - Markov process
  - Gaussian process
  - Lévy process (independent stationary increments)
- BM allows explicit calculations, which are impossible for more general objects
- BM can be used as a building block for other processes
- BM has many beautiful mathematical properties

Brownian motion. Definition

Def. **Brownian motion** with diffusion coefficient σ^2 is a continuous time stochastic process $(B_t)_{t\geq 0}$ satisfying

(i)

(ii)

(iii)

$\sigma^2 = 1 \leftarrow$ standard BM