MATH180C: Introduction to Stochastic Processes II

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Today: Martingales

Next: PK 8.1

Week 8:

Martingales

Definition. A stochastic process $(X_n, n \geq 0)$ is a martingale if for $n = 0, 1, \ldots$

(a) $E(|X_n|) < \infty$

(b) $E(X_{n+1} \mid X_0, X_1, \ldots, X_n) = X_n$

Thm. Let $(X_n)_{n\geq 0}$ be a martingale with nonnegative values. For any $\lambda > 0$ and $m \in \mathbb{N}$

$$P\left(\max_{0 \leq n \leq m} X_n \geq \lambda\right) \leq \frac{E(X_0)}{\lambda} \qquad (1)$$

and

$$P\left(\max_{n \geq 0} X_n \geq \lambda\right) \leq \frac{E(X_0)}{\lambda} \qquad (2)$$

Example

A gambler begins with a unit amount of money and faces a series of independent fair games. In each game the gambler bets fraction p of his current fortune, wins with probability $\frac{1}{2}$, loses with probability $\frac{1}{2}$. Estimate the probability that the gambler ever doubles the initial fortune.

Denote by $Z_n, n \geq 0$, the gambler's fortune after n-th game.

Denote $\{Y_i\}_{i=1}^{\infty}$ i.i.d. with $P(Y_i = 1+p) = P(Y_i = 1-p) = \frac{1}{2}$

Then $Z_n = Y_1 \cdot Y_2 \cdot \dots \cdot Y_n$

$E(Y_i) = 1 \Rightarrow (Z_n)$ is a martingale, nonnegative

$$\Rightarrow P\left(\max_{n \geq 0} Z_n \geq 2\right) \leq \frac{E(Z_0)}{2} = \frac{1}{2}$$

Martingale transform

In the previous example the stake in n-th game is $p Z_{n-1}$. What if we choose another strategy?

<u>Def</u> Let $(X_n)_{n\geq 0}$ be a nonnegative martingale, and let $(C_n)_{n\geq 0}$ be a stochastic process with $C_n = f_n(X_0, \dots, X_{n-1})$. Then the stochastic process

$$Z_n := \sum_{k=1}^{n} C_k (X_k - X_{k-1}) =: (C \bullet X)_n \,, \quad (C \bullet X)_0 = 0$$

is called the martingale transform of X by C

Think of
- $X_k - X_{k-1}$ as the winning per unit stake in k-th game
- C_k as your stake in k-th game
  decision is made based on the previous history
- $(C \bullet X)_n$ as total winnings up to time n

Martingale transform

Prop. Let $Z_n = X_0 + (C \cdot X)_n$. Let $C_k > 0$ bounded if $Z_{k-1} > 0$ and $C_k = 0$ if $Z_{k-1} = 0$. Then $(Z_n)_{n \geq 0}$ is a martingale

Proof: $E(Z_{n+1} \mid Z_0, \dots, Z_n) = E(Z_n + C_{n+1}(X_{n+1} - X_n) \mid Z_0, \dots, Z_n)$

$$= Z_n + E(C_{n+1}(X_{n+1} - X_n) \mid Z_0, \dots, Z_n)$$

Note that $Z_n - Z_{n-1} = C_n (X_n - X_{n-1})$

If $Z_n > 0$, then $C_1 > 0, \dots, C_n > 0$,

$$X_1 = (Z_1 - Z_0) C_1^{-1} + Z_0, \quad X_n = (Z_n - Z_{n-1}) C_n^{-1} + X_{n-1} \quad \text{and}$$

$$E(Z_{n+1} \mid Z_0, \dots, Z_n) = Z_n + E(C_{n+1}(X_{n+1} - X_n) \mid X_0, \dots, X_n)$$

$$= Z_n + C_{n+1}\Big(\underbrace{E(X_{n+1} \mid X_0, \dots, X_n)}_{= X_n} - X_n\Big) = Z_n$$

If $Z_n = 0$, then $C_{n+1} = 0$ and $E(Z_{n+1} \mid Z_0, \dots, Z_n) = 0 = Z_n$

Gambling example

Start from the initial fortune $X_0 = 1$.

Define $$Z_n = 1 + (C \bullet X)_n \geq 0$$

fortune after the n-th game with strategy C.

Then (Z_n) is a nonnegative martingale, $E(Z_0) = 1$

$$\Rightarrow \quad P\left(\max_{n \geq 0} Z_n \geq 2 \right) \leq \frac{1}{2}$$

Convergence of nonnegative martingales

Thm.

If $(X_n)_{n\geq 0}$ is a nonnegative (super)martingale, then with probability 1

$$\exists \lim_{n\to\infty} X_n =: X_\infty$$

and

$$E(X_\infty) \leq E(X_0)$$

Example

An urn initially contains one red ball and one green ball. Choose a ball and return it to the urn together with another ball of the same color. Repeat. Denote by X_n the fraction of red ball after n iterations.

Example (cont.)

(i) $(X_n)_{n\geq 0}$ is a martingale

Denote by R_n the number of red balls after n-th iteration

$$R_n = X_n(n+2)$$

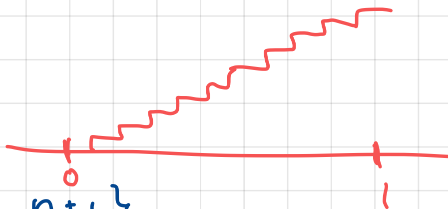
Then

$$E(X_{n+1} \mid X_0, \dots, X_n) = X_n \cdot \frac{R_n+1}{n+3} + (1-X_n)\cdot \frac{R_n}{n+3}$$

$$= \frac{1}{n+3}\left(\cancel{X_n \cdot R_n} + X_n + R_n - \cancel{X_n R_n}\right) = \frac{1}{\cancel{n+3}} X_n \cancel{(n+3)}$$

(ii) X_n is nonnegative $\Rightarrow$ $\exists \lim\limits_{n\to\infty} X_n =: X_\infty$

(iii) Compute the distribution of X_∞



$$P\left(X_n = \frac{k}{n+2}\right) = \frac{1}{n+1} \quad \text{for} \quad k \in \{1, 2, \dots, n+1\}$$

$$P(X_\infty \leq x) = x, \quad x \in (0,1) \Rightarrow X_\infty \sim \mathrm{Unif}([0,1])$$