MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Asymptotic behavior of renewal processes

Next: PK 2.5, Durrett 5.1-5.2

Week 7:

- HW6 due Monday, May 22 on Gradescope
- Midterm 2 on Wednesday, May 24

Example: Age replacement policies (PK, p. 363)

X_i - lifetime of i-th component, $F_{X_i}(t) = F(t)$

Y_i - times between failures

$N(t) = \#$ replacements on $[0,t]$, $Q(t) = \#$ failure replacements on $[0,t]$

Last time:

$$\frac{E(N(t))}{t} \approx \frac{1}{\mu_T} \quad \text{for large } t$$

$Q(t)$ renewal process with interrenewal times Y_i and

$$Y_1 = L \cdot T + Z \quad \text{with} \quad P(L \geq n) = (1 - F(T))^n, \; P(Z \leq z) = \frac{F(z)}{F(T)}$$

Example: Age replacement policies (PK, p. 363)

Now we can compute the long-run rate of the replacements due to failures

$$E(Y_1) = T\,E(L) + E(Z)$$

$$E(L) = \sum_{n=1}^{\infty} P(L \geq n) = \frac{1-F(T)}{F(T)} \, .$$

$$E(Z) = \frac{\int_0^T F(T) - F(x)\,dx}{F(T)} \, , \quad \text{so} \quad = \frac{1}{F(T)}\left(T - \int_0^T F(x)\,dx\right) = \frac{1}{F(T)} \int_0^T (1-F(x))\,dx$$

$$E(Y_1) = \frac{1}{F(T)}\left(T(1-\cancel{F(T)}) + \int_0^T (\cancel{F(T)} - F(x))\,dx\right) = \frac{\mu_T}{F(T)}$$

Applying the elementary renewal theorem to $Q(t)$

$$\frac{E(Q(t))}{t} \approx \frac{F(T)}{\mu_T} \quad \text{for large } t$$

Example: Age replacement policies (PK, p. 363)

Suppose that the cost of one replacement is K, and each replacement due to a failure costs additional c. Then, in the long run the total amount spent on the replacements of the component per unit of time is given by

$$C(T) \approx K \cdot \frac{1}{\mu_T} + c \cdot \frac{F(T)}{\mu_T} = \frac{K + c \cdot F(T)}{\int_0^T (1 - F(x))\,dx}$$

If we are given c, K and the distribution of the component's lifetime F, we can try to minimize the overall costs by choosing the optimal value of T.

Example: Age replacement policies (PK, p. 363)

For example, if $K=1$, $c=4$ and $X_1 \sim \mathrm{Unif}[0,1]$ $\left(F(x) = x \mathbb{1}_{[0,1]} \right)$

For $T \in [0,1]$, $\mu_T = \int_0^T (1-x)\,dx = T\left(1-\frac{T}{2}\right)$ and

the average (per unit of time) long-run costs are

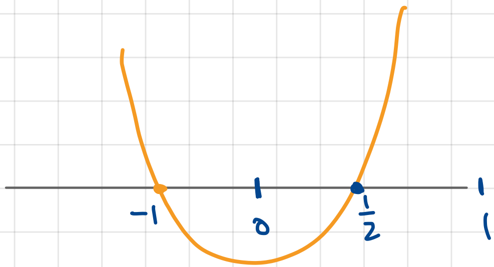
$$C(T) = \frac{1+4T}{T\left(1-\frac{T}{2}\right)}$$

$$\frac{d}{dT} C(T) = \frac{2T^2+T-1}{\left(T\left(1-\frac{T}{2}\right)\right)^2} = 0 \qquad T_1 = -1, \quad T = \frac{1}{2}$$

$T_{min} = \frac{1}{2}$

$C(T_{min}) = 8$

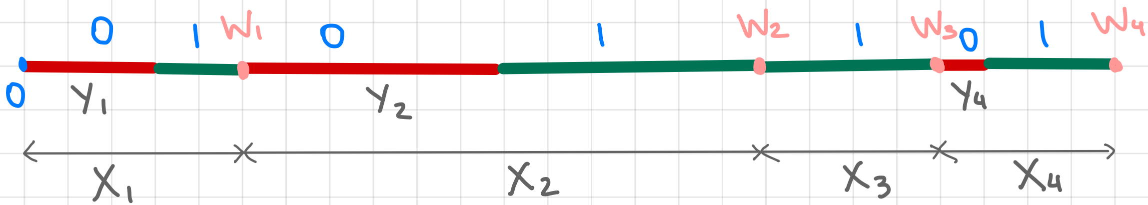
$C(1) = 10 > 8$



Two component renewals

Consider the following model:

- $(X_i)_{i=1}^{\infty}$ are interrenewal times
- at each moment of time the system $S(t)$ can be in one of two states: $S(t)=0$ or $S(t)=1$
- random variables Y_i denote the part of X_i during which the system is in state 0, $0 \le Y_i \le X_i$
- collection $((X_i, Y_i))_{i=1}^{\infty}$ is i.i.d.



Q: In the long run (for large t), what is the probability that the system is in state 0 at time t?

Two component renewals

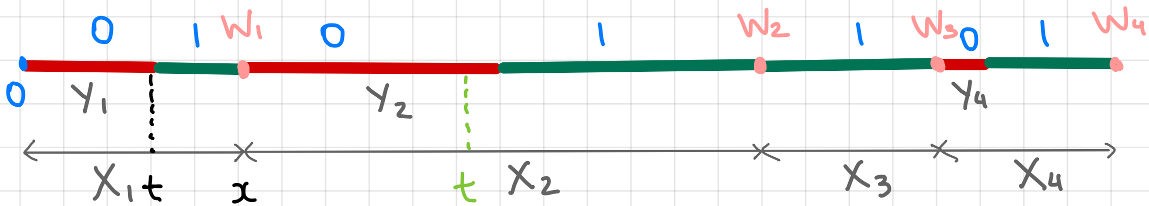
Thm. If $E(X_i) < \infty$, then $\lim_{t\to\infty} P(S(t)=0) = \dfrac{E(Y_1)}{E(X_1)}$

Proof. Denote $g(t) = P(S(t)=0)$. Then

$$g(t) = \int_0^\infty P(S(t)=0 \mid X_1 = x)\, dF(x)$$

If $t < x$, then $P(S(t)=0 \mid X_1 = x) = P(Y_1 > t \mid X_1 = x)$

If $t \geq x$, then $P(S(t)=0 \mid X_1 = x) = P\big(S(t-x)=0\big) = g(t-x)$



Two component renewals

$$g(t) = \underbrace{\int_t^\infty P(Y_1 > t \mid X_1 = x)\,dF(x)}_{h(t)} + \underbrace{\int_0^t g(t-x)\,dF(x)}_{g * F(t)}$$

Function g satisfies the renewal equation

$$g(t) = h(t) + g * F(t)$$

Note that $Y_1 \le X_1$, therefore $P(Y_1 > t \mid X_1 = x) = 0$ for $x < t$,

$$h(t) = \int_0^\infty P(Y_1 > t \mid X_1 = x)\,dF(x) = P(Y_1 > t)$$

$$\int_0^\infty h(t)\,dt = \int_0^\infty P(Y_1 > t)\,dt = E(Y_1)$$

From the Key renewal theorem $\lim_{t\to\infty} g(t) = \dfrac{\int_0^\infty h(t)\,dt}{E(X_1)} = \dfrac{E(Y_1)}{E(X_1)}$

Example: the Peter principle

Setting:
- infinite population of candidates for certain position
- fraction p of the candidates are competent, $q = 1 - p$ are incompetent
- if a competent person is chosen, after time C_i this person gets promoted
- if an incompetent person is chosen, this person remains in the job until retirement (r.v. I_j)
- once the position is open again, the process repeats

Question: What fraction of time, denoted f, is the position held by an incompetent person on average in the long run?

Example: the Peter principle

Denote $X_i = \begin{cases} C_i, & \text{if occupied by a competent person} \\ I_i, & \text{if occupied by an incompetent person} \end{cases}$

$Y_i = \begin{cases} 0, & \text{if occupied by a competent person} \\ I_i, & \text{if occupied by an incompetent person} \end{cases}$

KRT for two component renewals can be applied to $((X_i, Y_i))_{i=1}^{\infty}$

If $S(t) = 0$ if the person is incompetent, then

$$\lim_{t \to \infty} P(S(t)=0) = \frac{E(Y_1)}{E(X_1)} \qquad \text{and}$$

$$f := \lim_{t\to\infty} \left(\frac{1}{t} E\left[\int_0^t \mathbb{1}_{\{S(u)=0\}} \, du \right] \right) = \lim_{t \to 0} \frac{1}{t} \int_0^t P(S(u)=0) \, du = \frac{E(Y)}{E(X)}$$

exercise

Finally, if
- $E(C_i) = \mu$
- $E(I_i) = \nu$

, then $f = \dfrac{E(Y_1)}{E(X_1)} = \dfrac{(1-p)\nu}{p\mu + (1-p)\nu}$

Example: the Peter principle

If we take $p = \frac{1}{2}$, $\mu = 1$, $\nu = 10$, then

$$f = \frac{\frac{1}{2} \cdot 10}{\frac{1}{2} \cdot 1 + \frac{1}{2} 10} = \frac{10}{11} = 0.909$$