MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Asymptotic behavior of renewal processes

Next: PK 7.5, Durrett 3.1, 3.3

Week 6:

- HW5 due Friday, May 12 on Gradescope

Key renewal theorem

Thm (Key renewal theorem) Let h be locally bounded.

(a) If H satisfies $H = h + h * M$, then H is locally bounded and

$$H = h + H * F \quad (*)$$

(b) Conversely, if H is a locally bounded solution to $(*)$, then

$$H = h + h * M \quad (**)$$

[convolution in the Riemann–Stieltjes sense]

(c) If h is absolutely integrable, then

$$\lim_{t \to \infty} H(t) = \frac{\int_0^\infty h(x)\,dx}{\mu}$$

No proof.

Remark. Key renewal theorem says that if h is locally bounded, then there exists a unique locally bounded solution to $(*)$ given by $(**)$

Important remark

Let $W = (W_1, W_2, \ldots)$ be renewal times of a renewal process, and denote $W' = (W_1', W_2', \ldots)$ with

$$W_i' = W_{i+1} - W_1 = X_2 + X_3 + \cdots + X_{i+1},$$

shifted arrival times.

Then:

- W' is independent of $W_1 = X$,
- W' has the same distribution as W

Example

Example. Compute $\lim_{t\to\infty} E(\gamma_t)$. Take $H(t) = E(\gamma_t)$

If $X_1 > t$, then $\gamma_t = X_1 - t$; if $X_1 < t$ condition on $X_1 = s$

$$E(\gamma_t) = E(\gamma_t \mathbb{1}_{X_1 > t}) + E(\gamma_t \mathbb{1}_{X_1 \leq t})$$

$$E(\gamma_t \mathbb{1}_{X_1 \leq t}) =$$

$$=$$

$$=$$

$$=$$

$$=$$

Example (cont)

Assume that $E(X_1) = \mu$, $Var(X_1) = \sigma^2$

$$E((X_1 - t)\mathbb{1}_{X_1 > t}) =$$

$$=$$

Since we assume that $E(X_1) = \sigma^2$,

and

Finally, we have that

$$H(t) =$$

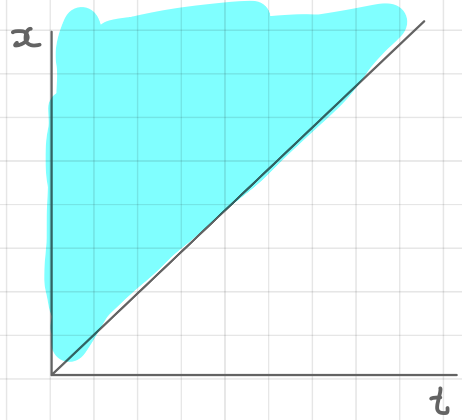
therefore $H(t) = h(t) + h * M(t)$

with $h(t) =$

Example (cont)

In particular,

$$\int_0^\infty \int_t^\infty (1-F(x))\,dx\,dt =$$



$\Rightarrow$ by part (c) of the key renewal theorem

$$\lim_{t\to\infty} E(\gamma_t) =$$

Similarly $\lim_{t\to\infty} E(\delta_t) =$, $\lim_{t\to\infty} E(\beta_t) =$

Example

What is the expected time to the next earthquake in the long run?

For $X_1 \sim \text{Unif}[0,1]$

therefore, $\lim_{t \to \infty} E(\gamma_t) =$

And the long run expected time between two consecutive earthquakes is

Remark: moments of nonnegative r.v.s

<u>Proposition.</u> Let X be a nonnegative random variable. Then

$$E(X^n) =$$

$$=$$

<u>Proof.</u> $X \geq 0 \Rightarrow X^n \geq 0$. Using the "tail" formula for the expectation of nonnegative random variables

$$E(X^n) =$$

After the change of variable $x = t^{1/n}$ we get

$$E(X^n) =$$

Remark. $M(t)$ is finite for all t

Proposition. Let $N(t)$ be a renewal process with interrenewal times X_i having distribution F. If there exist $c>0$ and $\alpha \in (0,1)$ such that $P(X_1 > c) > \alpha$, then

Proof: Recall that $M(t) = \sum_{k=1}^{\infty} P(W_k \leq t) = \sum_{k=1}^{\infty} P\left(\sum_{j=1}^{k} X_j \leq t\right)$ $(*)$

Example: Age replacement policies (PK, p. 363)

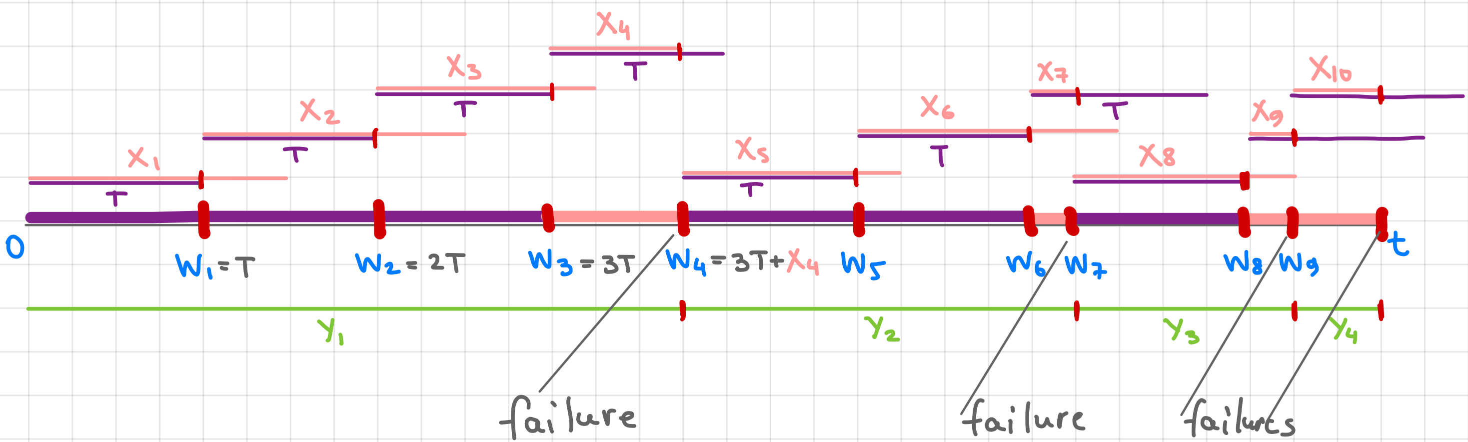
Setting: - component's lifetime has distribution function F
- component is replaced
  (A) either when it fails,
  (B) or after reaching age T (fixed)
  whichever occurs first
- replacements (A) and (B) have different costs:
  replacement of a failed component (A) is more expensive than the planned replacement (B)

Question: How does the long-run cost of replacement depend on the cost of (A), (B) and age T?
What is the optimal T that minimizes the long-run cost of replacement?

Example: Age replacement policies (PK, p. 363)

Notation: X_i - lifetime of i-th component, $F_{X_i}(t) = F(t)$

Y_i - times between failures



Here we have two renewal processes

(1) renewal process $N(t)$ generated by renewal times $(W_i)_{i=1}^{\infty}$

(2) renewal process $Q(t)$ generated by interrenewal times $(Y_i)_{i=1}^{\infty}$

$N(t) =$, $Q(t) =$

Example: Age replacement policies (PK, p. 363)

Compute the distribution of the interrenewal times for $N(t)$

$$W_i - W_{i-1} = \begin{cases} & \\ & \end{cases}, \text{ so}$$

$$F_T(x) := P(W_i - W_{i-1} \leq x) = \begin{cases} & \\ & \end{cases}$$

In particular,

$$E(W_i - W_{i-1}) =$$

Using the elementary renewal theorem for $N(t)$, the total number of replacements has a long-run rate

Example: Age replacement policies (PK, p. 363)

Compute the distribution of the interrenewal times for $\odot(t)$.

$$
Y_1 = \begin{cases} & \text{if } X_1 \le T \\ & \text{if } X_1 > T,\ X_2 \le T \\ \vdots & \\ & ,\ \text{if } X_1 > T, \dots, X_n > T,\ X_{n+1} \le T \\ \vdots & \end{cases}
$$

so

and for $z \in (0, T)$

$$P(Z \le z) =$$

$$=$$

$$=$$

Example: Age replacement policies (PK, p. 363)

Now we can compute the long-run rate of the replacements due to failures

$E(Y_1) =$

$E(L) =$

$E(Z) =$, so

$E(Y_1) =$

Applying the elementary renewal theorem to $Q(t)$

Example: Age replacement policies (PK, p. 363)

Suppose that the cost of one replacement is K, and each replacement due to a failure costs additional c. Then, in the long run the total amount spent on the replacements of the component per unit of time is given by

$$C(T) \approx$$

If we are given c, K and the distribution of the component's lifetime F, we can try to minimize the overall costs by choosing the optimal value of T.

Example: Age replacement policies (PK, p. 363)

For example, if $K=1$, $C=4$ and $X_1 \sim \mathrm{Unif}[0,1]$ ($F(x) = x \mathbb{1}_{[0,1]}$)

For $T \in [0,1]$, $\mu_T =$ and

the average (per unit of time) long-run costs are

$$C(T) =$$

$$\frac{d}{dT} C(T) =$$