MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Asymptotic behavior of renewal processes

Next: PK 7.5, Durrett 3.1, 3.3

Week 6:

- HW5 due Friday, May 12 on Gradescope

Asymptotic behavior of renewal processes

Let $N(t)$ be a renewal process with interrenewal times X_i, $X_i \in (0, \infty)$.

<u>Thm</u>.

$$P\left(\lim_{t\to\infty} N(t) = +\infty \right) = 1$$

<u>Proof</u>. $N(t)$ is nondecreasing, therefore $\exists \lim\limits_{t\to\infty} N(t) =: \underset{\substack{\Downarrow \\ (0,+\infty)\cup\{+\infty\}}}{N_\infty}$

N_∞ is the total number of events ever happened.

$N_\infty \le k$ if and only if $W_{k+1} = \infty$

if and only if $X_i = \infty$ for some $1 \le i \le k+1$

$$P(N_\infty < \infty) = P(X_i = \infty \text{ for some } 1 \le i \le k+1) = P\left(\bigcup_{i=1}^{k+1} \{X_i = \infty\}\right) \le \sum_{i=1}^{k+1} P(X_i = \infty) = 0$$

<u>Thm</u> (Pointwise renewal thm).

$$P\left(\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} \right) = 1 \qquad \mu = E(X_i)$$

Elementary Renewal Theorem

Thm. If $M(t) = E(N(t))$ and $E(X_1) = \mu$, then

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{\mu}$$

Proof. (Only for bounded X_i: $\exists K$ s.t. $P(X_1 \le K) = 1$)

First note that $W_{N(t)+1} = t + \gamma_t$

In lecture 13 we showed that $E(W_{N(t)+1}) = \mu(M(t)+1)$,

so
$$M(t) = \frac{t + E(\gamma_t)}{\mu} - 1$$

$$\frac{M(t)}{t} = \frac{1}{\mu} + \frac{1}{t}\left(\frac{E(\gamma_t)}{\mu} - 1\right) \longrightarrow \frac{1}{\mu} \quad \text{as } t\to\infty$$

If $X_i \le K$, then $\gamma_t \le K \Rightarrow E(\gamma_t) \le K$ $\blacksquare$

Ex: $(X_n)_{n\ge 0}$: 1) $P(\lim_{n\to\infty} X_n = 0) = 1$ 2) $\lim_{n\to\infty} E(X_n) > 0$

Asymptotic distribution of $N(t)$

<u>Thm</u>. Let $N(t)$ be a renewal process with $E(X_1)=\mu$ and $Var(X_1)=\sigma^2$, then

1)
$$\lim_{t\to\infty}\frac{Var(N(t))}{t}=\frac{\sigma^2}{\mu^3}$$

2)
$$\lim_{t\to\infty}P\left(\frac{N(t)-E(N(t))}{\sqrt{Var(N(t))}}\le x\right)=\lim_{t\to\infty}P\left(\frac{N(t)-\frac{t}{\mu}}{\sqrt{\frac{\sigma^2}{\mu^3}t}}\le x\right)$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{y^2}{2}}\,dy$$

No proof.

$$N(t)\approx\frac{t}{\mu}+\sqrt{\frac{\sigma^2}{\mu^3}t}\cdot Z,$$ where $Z\sim N(0,1)$ for t large

Elementary renewal theorem and continuous X_i's

Two more results (without proofs) about the limiting behaviour of $M(t)$ for models with continuous interrenewal times.

<u>Thm</u>. Let $E(X_1) = \mu$ and let $m(t) = \frac{d}{dt} M(t)$ be the renewal density. Then

$$\lim_{t \to \infty} m(t) = \lim_{t \to \infty} \frac{dM(t)}{dt} = \frac{1}{\mu}$$

<u>Remark</u> $\lim_{t \to \infty} \frac{f(t)}{t} = \alpha$ does not imply in general $\lim_{t \to \infty} f'(t) = \alpha$

(E.g., take $f(t) = t + \sin(t)$)

<u>Thm</u>. If additionally $\mathrm{Var}(X_1) = \sigma^2$, then

$$\lim_{t \to \infty} \left(M(t) - \frac{t}{\mu} \right) = \frac{\sigma^2 - \mu^2}{2\mu^2}$$

Example: $X_i \sim$ Gamma$(2,1)$

Let $N(t)$ be a renewal process with interrenewal times X_i having Gamma distribution with parameters $(2,1)$ i.e., $f_{X_1}(t) = t\,e^{-t}$. Then from the properties of the Gamma distribution (or from direct computations)

$$X_1 + \cdots + X_n \sim \text{Gamma}(2n,1), \text{ so}$$

$$f^{*n}(t) = \frac{t^{2n-1}}{(2n-1)!} e^{-t}, \quad \text{for } t>0$$

We can compute the renewal density

$$m(t) = \sum_{n=1}^{\infty} f^{*n}(t) = \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!} e^{-t} = \frac{e^{t}-e^{-t}}{2} e^{-t} = \frac{1-e^{-2t}}{2}$$

so that $M(t) = \int_0^t m(x)\,dx = \int_0^t \frac{1-e^{-2x}}{2}\,dx = \frac{t}{2} - \frac{1}{4} + \frac{1}{4}e^{-2t}$

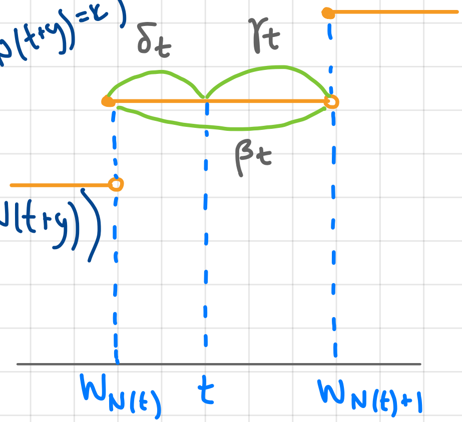
Finally, $E(X_1) = \mu = 2$, $Var(X_1) = \sigma^2 = 2$, so $\frac{\sigma^2 - \mu^2}{2\mu^2} = \frac{2-4}{2\cdot 4} = -\frac{1}{4}$

Joint distribution of age and excess life

From the definition of γ_t and δ_t

$$P(\delta_t \geq x, \gamma_t > y) \qquad (x \leq t)$$

$$= P(W_{N(t)} \leq t-x,\ W_{N(t)+1} > t+y) = P(N(t-x) = N(t+y)) = \sum_{k=0}^{\infty} P(N(t-x) = N(t+y) = k)$$



- Partition wrt the values of $N(t)$

$$= \sum_{k=0}^{\infty} P(W_k \leq t-x,\ W_{k+1} > t+y)$$

condition on the value of W_k (c.d.f. of W_k is $F^{*k}(t)$)

$$= \underbrace{1 - F(t+y)}_{P(W_1 > t+y)} + \sum_{k=1}^{\infty} \int_0^{\infty} P(\underbrace{W_k \leq t-x}_{u \leq t-x},\ \underbrace{W_{k+1}}_{W_k + X_{k+1}} > t+y \mid W_k = u)\, dF^{*k}(u)$$

$$= 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{t-x} P(u + X_{k+1} > t+y)\, dF^{*k}(u)$$

$$= 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{t-x} (1 - F(t+y-u))\, dF^{*k}(u)$$

Limiting distribution of age and excess life

Assume that X_i are continuous. Then

$$P(\delta_t \geq x, \gamma_t > y) = 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{t-x} (1 - F(t+y-u)) \, dF^{*k}(u)$$

$$= 1 - F(t+y) + \int_0^{t-x} (1 - F(t+y-u)) \, d \overbrace{\sum_{k=1}^{\infty} F^{*k}(u)}^{M(t)}$$

$$= 1 - F(t+y) + \int_0^{t-x} (1 - F(t+y-u)) \, m(u) \, du$$

$$=$$

Recall that $\varepsilon(s) := m(s) - \frac{1}{\mu} \to 0$ as $s \to \infty$ $(\mu = E(X_1))$. Then

$$\lim_{t \to \infty} P(\delta_t \geq x, \gamma_t > y) =$$

$$=$$