MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Poisson process as a renewal process

Next: PK 7.2-7.3, Durrett 3.1

Week 5:

- HW4 due Friday, May 5 on Gradescope

Expectation of W_n

Proposition 2. Let $N(t)$ be a renewal process with interrenewal times $X_1, X_2, \ldots$ and renewal times $(W_n)_{n\geq 1}$. Then

$$E(W_{N(t)+1}) = E(X_1)\ E(N(t)+1)$$
$$= \mu(M(t)+1)$$

where $\mu = E(X_1)$.

Proof. $E(W_{N(t)+1}) = E(X_1 + X_2 + \cdots + X_{N(t)+1}) = \underset{\mu}{\underset{\shortparallel}{E(X_1)}} + E(X_2 + \cdots + X_{N(t)+1})$

$$E(X_2 + \cdots + X_{N(t)+1}) = E(X_2 \mid N(t)=1)\,P(N(t)=1)$$
$$+ E(X_2 + X_3 \mid N(t)=2)\,P(N(t)=2)$$
$$+ E(X_2 + X_3 + X_4 \mid N(t)=3)\,P(N(t)=3)$$
$$\vdots$$
$$+ E\left(\sum_{k=2}^{n+1} X_k \mid N(t)=n\right) P(N(t)=n) + \cdots$$

$$= \sum_{n=1}^{\infty} E(X_2 \mid N(t)=n)\,P(N(t)=n) + \sum_{n=2}^{\infty} E(X_3 \mid N(t)=n)\,P(N(t)=n) + \cdots$$

Renewal equation

Proposition 3. Let $(N(t))_{t\geq 0}$ be a renewal process with interrenewal distribution F. Then $M(t) = E(N(t))$ satisfies

$$M(t) = F(t) + M * F(t) = F(t) + \int_0^t M(t-x)\, dF(x)$$

renewal equation

Proof. We showed in Proposition 1 that

$$M = \sum_{n=1}^{\infty} F^{*n}.$$

Then

$$M * F = \left(\sum_{n=1}^{\infty} F^{*n}\right) * F = \sum_{n=2}^{\infty} F^{*n} = \sum_{n=1}^{\infty} F^{*n} - F$$

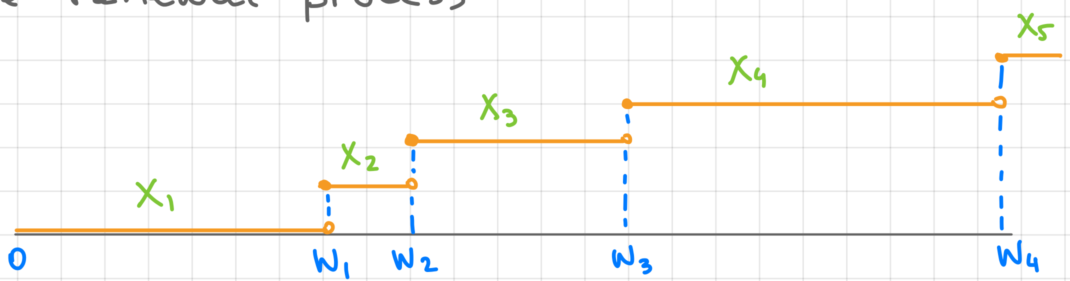
$$= M - F$$

∎

Poisson process as a renewal process

The Poisson process $N(t)$ with rate $\lambda > 0$ is a renewal process with $F(x) = 1 - e^{-\lambda x}$.

- sojourn times S_i are i.i.d., $S_i \sim Exp(\lambda)$
- S_i represent intervals between two consecutive events (arrivals of customers)
- $W_n = \sum_{i=0}^{n-1} S_i$
- we can take $X_i = S_{i-1}$ in the definition of the renewal process



Poisson process as a renewal process

$$= \int G(t-x)\, dF(x)$$

$$F'(x)dx$$

We know that $N(t) \sim \text{Pois}(\lambda t)$, so in particular

$$G * F(t) = \int G(t-x) f(x) dx$$

$$E(N(t)) = \lambda t$$

Example Compute $M(t) = \sum_{n=1}^{\infty} F^{*n}(t)$ for PP

$$F_2(t) = \int_0^t \Big(\underbrace{1 - e^{-\lambda(t-x)}}_{F(t-x)}\Big) \underbrace{\lambda e^{-\lambda x}}_{f(x)} dx = \underbrace{1 - e^{-\lambda t}}_{1-\varphi_0(t)} - \lambda \int_0^t e^{-\lambda(t-x)} e^{-\lambda x} dx = F(t) - \underbrace{\lambda t e^{-\lambda t}}_{\varphi_1(t)}$$

Denote $\varphi_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$:

$$\varphi_k * F(t) = \int_0^t \frac{(\lambda(t-x))^k}{k!} e^{-\lambda(t-x)} \lambda e^{-\lambda x} dx = \frac{\lambda^{k+1}}{k!} e^{-\lambda t} \int_0^t (t-x)^k dx$$

$$= \frac{\lambda^{k+1} t^{k+1}}{(k+1)!} e^{-\lambda t} = \varphi_{k+1}(t)$$

$$F * F(t) = F(t) - \varphi_1(t)$$

$$F^{*3}(t) = (F - \varphi_1) * F(t) = F * F - \varphi_1 * F = F - \varphi_1 - \varphi_2$$

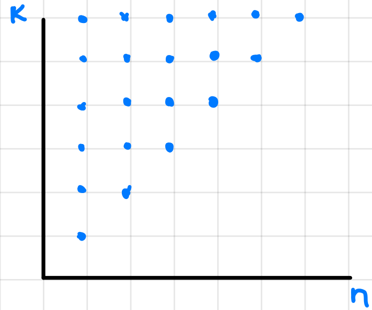
$$\vdots$$

$$F^{*n}(t) = F - \varphi_1 - \varphi_2 - \cdots - \varphi_{n-1}$$

Poisson process as a renewal process (cont.)

$$\sum_{n=1}^{\infty} F^{*n}(t) = \sum_{n=1}^{\infty}\left(1-\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}e^{-\lambda t}\right) = e^{-\lambda t}\sum_{n=1}^{\infty}\left(e^{\lambda t}-\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}\right)$$

$$= e^{-\lambda t}\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}-\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}\right)$$



$$= e^{-\lambda t}\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\frac{(\lambda t)^k}{k!}$$

$$= e^{-\lambda t}\sum_{k=1}^{\infty}\sum_{n=1}^{k}\frac{(\lambda t)^k}{k!}$$

$$= e^{-\lambda t}\sum_{k=1}^{\infty}\frac{(\lambda t)^k}{(k-1)!} = \lambda t\, e^{-\lambda t}\sum_{k=1}^{\infty}\frac{(\lambda t)^{k-1}}{(k-1)!} = \lambda t$$

$$M(t) = \lambda t$$

Renewal density

$$g * f(t) = \int_{-\infty}^{\infty} g(t-x) f(x)\, dx$$

<u>Proposition</u> Let $N(t)$ be a renewal process with continuous interrenewal times X_i having density $f(x)$. Denote $m(t) = \sum_{n=1}^{\infty} f^{*n}(t)$. Then $M(t) = \int_0^t m(x)\, dx$

and $m(t) = f(t) + m * f(t)$ (*)

renewal density

<u>Proof</u>: $\frac{d}{dt} F^{*n}(t) = \left(\frac{d}{dt} F^{*(n-1)}\right) * f(t) = f^{*n}(t)$ ■

<u>Example</u>: Compute the renewal density for PP using (*).

$f(x) = \lambda e^{-\lambda x}$, so (*) becomes

$$m(t) = \lambda e^{-\lambda t} + \int_0^t m(t-x) \lambda e^{-\lambda x} dx = \lambda e^{-\lambda t} + \int_0^t m(x) \lambda e^{-\lambda (t-x)} dx$$

$$= \lambda e^{-\lambda t} \left(1 + \int_0^t m(x) e^{\lambda x} dx\right)$$

(cont.)

$$e^{\lambda t} m(t) = \lambda\left(1 + \int_0^t e^{\lambda x} m(x)\,dx\right) \leftarrow \text{differentiate}$$

$$\left\{\begin{array}{l} \dfrac{d}{dt}\left(e^{\lambda t} m(t)\right) = \lambda e^{\lambda t} m(t) \\ m(0) = \lambda \end{array}\right. \Longrightarrow \begin{array}{l} e^{\lambda t} m(t) = e^{\lambda t} \cdot C \\ m(t) = \lambda \end{array}$$

Indeed, $M(t) = \int_0^t \lambda\,dt = \lambda t$