MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: MC review. Conditioning on continuous random variables

Next: PK 7.1, Durrett 3.1

Week 4:

- HW3 due Saturday, April 29 on Gradescope
- Midterm 1: Friday, April 28

Remarks

4) Similarly as in the discrete case, π_j gives the fraction of time spent in state j in long run

$$\lim_{T \to \infty} \mathbb{E}\left[\frac{1}{T} \int_0^T \mathbb{1}_{\{X_t = j\}} \, dt \;\middle|\; X_0 = i\right] = \pi_j$$

(compare with $\lim\limits_{m \to \infty} E\left[\frac{1}{m} \sum_{n=0}^{m-1} \mathbb{1}_{\{X_n = j\}} \mid X_0 = i\right] = \pi_j$ for discrete time MC)

5) If we can find $(\pi_i)_{i=0}^{\infty}$ such that $\pi_i q_{ij} = \pi_j q_{ji}$ then $(\pi_i)_{i=0}^{\infty}$ satisfies $\pi Q = 0$

Indeed, $\sum_{j=0}^{\infty} \pi_i q_{ij} = \pi_i \sum_{j=0}^{\infty} q_{ij} = 0 = \sum_{j=0}^{\infty} \pi_j q_{ji} = (\pi Q)_i$

Example: Birth and death processes

If we consider the birth and death process, the equation $\pi Q = 0$

takes the following form

where $\theta_i = \frac{\lambda_{i-1}}{\mu_i} \cdot \frac{\lambda_{i-2}}{\mu_{i-1}} \cdot \dots \cdot \frac{\lambda_0}{\mu_1}$, $\theta_0 = 1$.

Then, $\sum_{i=0}^{\infty} \pi_i = 1$ implies that

If $\sum_{i=0}^{\infty} \theta_i < \infty$, then (X_t) is positive recurrent and $\pi_j =$

If $\sum_{i=0}^{\infty} \theta_i = \infty$, then $\pi_j = 0 \ \forall j$.

Example. Linear growth with immigration

Birth and death process, $\lambda_j = \lambda j + a$, $\mu_j = \mu j$ $(*)$

Using Kolmogorov's equations we showed (lecture 9) that $E(X_t) \to \frac{a}{\mu - \lambda}$, $t \to \infty$, if $\mu > \lambda$.

What is the limiting distribution of X_t?

From the previous slide, $\pi_j = \frac{\theta_j}{\sum_{i=0}^{\infty} \theta_i}$, $\theta_j = \frac{\lambda_{j-1} \cdots \lambda_0}{\mu_j \cdots \mu_1}$

If we replace λ_j, μ_j by $(*)$, we get

$$\pi_j = \left(\frac{\lambda}{\mu}\right)^j \left(1 - \frac{\lambda}{\mu}\right)^{a/\lambda} \frac{\frac{a}{\lambda}\left(\frac{a}{\lambda}+1\right)\cdots\left(\frac{a}{\lambda}+j-1\right)}{j!}, \quad j \geq 1$$

$$\pi_0 = \left(1 - \frac{\lambda}{\mu}\right)^{-\frac{a}{\lambda}}$$

What you should know for midterm 1 (minimum):

- definition of continuous time MC, Markov property, transition probabilities, generator
- representations of MC: infinitesimal (generator), jump-and-hold, transition probabilities, rate diagram and relations between them (in particular Q and $P(t)$)
- computing absorption probabilities and mean time to absorption
- computing stationary distributions for finite and infinite state MCs and interpretation of $(\pi_i)_{i=0}^{\infty}$
- basic properties of birth and death processes

Conditioning on continuous r.v.

Def. Let X and Y be jointly distributed continuous random variables with joint probability density function $f_{X,Y}(x,y)$. We call the function

the conditional probability density function of X given $Y=y$.

The function

is called conditional distribution of X given $Y=y$

Conditional expectation

Def. Let X and Y be jointly distributed continuous random variables, let $f_{X|Y}(x|y)$ be a conditional distribution of X given $Y=y$ and let $g: \mathbb{R} \to \mathbb{R}$ be a function for which $E(|g(X)|) < \infty$.

Then we call

the conditional expectation of $g(X)$ given $Y=y$.

In particular, if

Remark

If Y is a continuous random variable, then

Therefore, we cannot define $P(X \in A \mid Y=y)$ as

$$P(X \in A \mid Y=y) =$$

On the other hand consider example:

Intuitive explanation / derivation

$P(X \in [x, x+\Delta x], Y \in [y, y+\Delta y])$

$=$

Using the multiplication rule ($f_Y(y) > 0$ on $[y, y+\Delta y]$)

$P(X \in [x, x+\Delta x], Y \in [y, y+\Delta y])$

$=$

$P(X \in [x, x+\Delta x] \mid Y \in [y, y+\Delta y])$

Properties of conditional probability/expectation

1) $P(a < X < b, c < Y < d) =$

2) $P(a < X < b) =$

3) $E(g(X)) =$

Further properties of conditional expectation (PK, p.50)

4) $E(c_1 g_1(X_1) + c_2 g_2(X_2) \mid Y=y) = c_1 E(g_1(X_1) \mid Y=y) + c_2 E(g_2(X_2) \mid Y=y)$

5) $E(v(X,Y) \mid Y=y) =$

In particular, $E(v(X,Y)) =$

6) $E(g(X)h(Y)) = \int_{-\infty}^{+\infty} h(y) E(g(X) \mid Y=y) f_Y(y)\, dy$

$= E(h(Y) E(g(X) \mid Y))$

7) $E(g(X) \mid Y=y) = E(g(X))$ if X and Y are independent

Example 1

Let (X, Y) be jointly continuous r.v.s with density

$$f_{X,Y}(x,y) = \frac{1}{y} e^{-\frac{x}{y} - y}, \quad x, y > 0$$

Compute the conditional density of X given $Y=y$.

1) Compute the marginal density of Y

2) Compute the conditional density

Example 1 (cont.)

Suppose that $Y \sim Exp(1)$, and X has exponential distribution with parameter $\frac{1}{Y}$. Compute $E(X)$

$$E(X) = \int_0^\infty \int_0^\infty x \frac{1}{y} e^{-\frac{x}{y} - y} \, dx \, dy$$

Example 2: continuous and discrete r.v.s

Let $N \in \mathbb{N}$, $P \sim \mathrm{Unif}[0,1]$, $X \sim \mathrm{Bin}(N, P)$

What is the distribution of X?

$$P(X=k) =$$

$$=$$

$$=$$

$$=$$

Example 3

Let X and Y be i.i.d. $Exp(\lambda)$ r.v.

Define $Z = \frac{X}{Y}$. Compute the density of Z.

- If $X \sim Exp(\lambda)$, then for $\alpha > 0$ $\alpha X \sim Exp\left(\frac{\lambda}{\alpha}\right)$

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