MATH 180C HOMEWORK 6

SPRING 2023

Due date: **Friday 5/19/2023 11:59 PM** (via Gradescope)
Note that there are *Exercises* and *Problems* in the textbook. Make sure you read the homework carefully to find the assigned question.

1. *Pinsky and Karlin, Exercise 7.4.1.*
   Consider the triangular lifetime density $f(x) = 2x$ for $0 < x < 1$. Determine an asymptotic expression for the expected number of renewals up to time t.

   Solution. Let $M(t) = E(N(t))$ be the renewal function giving the expected number of renewals up to time t. From lecture 16

$$\lim_{t\to\infty}\left[M(t) - \frac{t}{\mu}\right] = \frac{\sigma^2 - \mu^2}{2\mu^2}, \tag{1}$$

   where μ and σ^2 are the mean and the variance of the interrenewal time. Compute μ and σ^2

$$\mu = \int_0^1 2x^2 dx = \frac{2}{3}, \quad \sigma^2 = \int_0^1 2x^3 dx - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}.$$

   Substituting these values into (1) gives the following asymptotic expression of $M(t)$ for large t

$$M(t) = \frac{3t}{2} + \left(\frac{1}{18} - \frac{4}{9}\right)\frac{9}{8} + o(1) = \frac{3t}{2} - \frac{7}{16} + o(1).$$

2. *Pinsky and Karlin, Exercise 7.4.2.*
   Consider the triangular lifetime density $f(x) = 2x$ for $0 < x < 1$. Determine an asymptotic expression for the probability distribution of excess life. Using this distribution, determine the limiting mean excess life.

   Solution. First compute the interrenewal distribution function

$$F(x) = \begin{cases} 0, & x < 0, \\ x^2, & 0 \le x < 1, \\ 1, & x \ge 1. \end{cases}$$

   The limiting density of the excess life is given by (see lecture 17)

$$f_{\gamma_\infty} = \frac{1}{\mu}(1 - F(x)) = \frac{3}{2}(1 - x^2)1_{[0,1]}(x), \tag{2}$$

where we used that $\mu = \int_0^1 2x^2 dx = 2/3$. Now we compute the limiting expected value of the excess life

$$\int_0^1 x \frac{3}{2}(1-x^2)dx = \frac{3}{2}\Big(\frac{1}{2} - \frac{1}{4}\Big) = \frac{3}{8}. \tag{3}$$

3. *Pinsky and Karlin, Exercise 7.4.4.*

Show that the optimal age replacement policy is to replace upon failure alone when lifetimes are exponentially distributed with parameter λ. Provide an intuitive explanation.

Solution. We start by writing the cost function from lecture 20

$$C(T) = \frac{K + cF(T)}{\int_0^T (1-F(x))dx} = \frac{K + c(1-e^{-\lambda T})}{\int_0^T e^{-\lambda x} dx} = \lambda \frac{K + c(1-e^{-\lambda T})}{1 - e^{-\lambda 6}} = \lambda\Big(\frac{K}{1-e^{-\lambda T}} + c\Big). \tag{4}$$

This function is monotonically decreasing, therefore the long-run costs are minimized when $T \to \infty$, i.e., when the component is replaced upon failure alone.

Intuitive explanation. Denote by X the lifetime of the component having exponential distribution. From the memoryless property of the exponential distribution we have

$$P(X > t + s | X > t) = P(X > s). \tag{5}$$

This means that if the component reaches certain age t, its lifetime distribution is the same as the lifetime distribution of a new component. In this case there is no advantage in replacing and 'old' component, since the probability that it fails is the same as for a new component.

4. *Pinsky and Karlin, Exercise 7.4.6.*

A machine can be in either of two states: 'up' and 'down'. It is up at time zero and thereafter alternates between being up and down. The lengths $X_1, X_2, \ldots$ of successive up times are independent and identically distributed random variables with mean α, and lengths $Y_1, Y_2, \ldots$ of successive down times are independent and identically distributed with mean β.

(a) In the long run, what fraction of time is the machine up?
Solution. The interrenewal times are $X_i + Y_i$ with $E(X_i + Y_i) = \alpha + \beta$, and X_i are the times that the machine is up with $E(X_i) = \alpha$. Denote by f the long-run fraction of time that the machine is up. Using the result from lecture 20,

$$f = \frac{E(X_i)}{E(X_i + Y_i)} = \frac{\alpha}{\alpha + \beta}. \tag{6}$$

(b) If the machine earns income at a rate of \$13 par unit time while up, what is the long run total rate of income earned by the machine?
Solution. From part (a), the fraction of time that the machine is up is $\frac{\alpha}{\alpha+\beta}$, thus per unit the machine earns $13\frac{\alpha}{\alpha+\beta}$ dollars.

(c) If each down time costs \$7, regardless of how long the machine is down, what is the long run total down time cost per unit time?

Solution. From the elementary renewal theorem, in the long run there are $\frac{1}{\alpha+\beta}$ switches to 'down' per unit of time. Therefore, the long run total down time cost per unit of time is $\frac{7}{\alpha+\beta}$ dollars.

5. *Pinsky and Karlin, Problem 7.4.1.*
Suppose that a renewal function has the form $M(t) = t + (1 - e^{-at})$. Determine the mean and variance of the interoccurrence distribution.

Solution. From the elementary renewal theorem we have

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{\mu}. \tag{7}$$

In our case,

$$\lim_{t\to\infty} \frac{M(t)}{t} = 1, \tag{8}$$

from which we conclude that $\mu = 1$. Furthermore, we know from lecture 16 that

$$\lim_{t\to\infty} \left[M(t) - \frac{t}{\mu}\right] = \frac{\sigma^2 - \mu^2}{2\mu^2}. \tag{9}$$

After subtracting t from $M(t)$ and taking the limit $t \to \infty$ we find

$$\frac{\sigma^2 - \mu^2}{2\mu^2} = 1. \tag{10}$$

Since we have already established that $\mu = 1$, we now compute

$$\frac{\sigma^2 - 1}{2} = 1, \tag{11}$$

which gives $\sigma^2 = 3$.

6. *Pinsky and Karlin, Problem 7.4.2.*
A system is subject to failures. Each failure requires a repair time that is exponentially distributed with rate parameter α. The operating time of the system until the next failure is exponentially distributed with rate parameter β. The repair times and the operating times are all statistically independent. Suppose that the system is operating at time 0. Determine an approximate expression for the mean number of failures up to time t, the approximation holding for $t \gg 0$.

Solution. Let $N(t)$ be the process that counts failures up to times t, and let $\tilde{N}(t)$ be the process that counts repairs (replacements) up to time t. Then $\tilde{N}(t)$ is a renewal process and

$$\tilde{N}(t) \le N(t) \le \tilde{N}(t) + 1. \tag{12}$$

The interrenewal times of $\tilde{N}(t)$ have mean $\mu := \frac{1}{\alpha} + \frac{1}{\beta}$ and variance $\sigma^2 := \frac{1}{\alpha^2} + \frac{1}{\beta^2}$.

From (12) and the elementary renewal theorem applied to $\tilde{N}(t)$ we get

$$\lim_{t\to\infty} \frac{E(N(t))}{t} = \lim_{t\to\infty} \frac{E(\tilde{N}(t))}{t} = \frac{1}{\mu} = \frac{\alpha\beta}{\alpha+\beta}. \tag{13}$$

Furthermore, using again (12) and the result from lecture 16

$$\lim_{t\to\infty}\Big(E(\tilde N(t)) - \frac{t}{\mu}\Big) = \frac{\sigma^2-\mu^2}{2\mu^2} \tag{14}$$

we get

$$\begin{aligned}
\frac{\sigma^2-\mu^2}{2\mu^2} &= \lim_{t\to\infty} E\Big(\tilde N(t) - \frac{t}{\mu}\Big) && (15)\\
&\le \lim_{t\to\infty} E\Big(N(t) - \frac{t}{\mu}\Big) && (16)\\
&\le \lim_{t\to\infty} E\Big(\tilde N(t) - \frac{t}{\mu}\Big) + 1 && (17)\\
&= \frac{\sigma^2-\mu^2}{2\mu^2} + 1, && (18)
\end{aligned}$$

which gives an approximation of $E(N(t))$. Notice that we have only found the bounds for the constant in the approximation, an interval of length 1 that contains the limit.

In order to improve the approximation, condition on the first failure time. Denote by $N'(t)$ the renewal process starting after the first failure and counting the failures that occur after the first failure. Then $N'(t)$ is a renewal process. Denote $M(t) := E(N'(t))$. Then

$$E(N(t)) = \int_0^t (1 + M(t-s))\beta e^{-\beta s} ds = 1 - e^{-\beta t} + \int_0^t M(t-s)\beta e^{-\beta s} ds. \tag{19}$$

From lecture 16 we have

$$\lim_{t\to\infty}\Big(M(t) - \frac{t}{\mu} - \frac{\sigma^2-\mu^2}{2\mu^2}\Big) = 0. \tag{20}$$

In particular, if we fix $\varepsilon > 0$, then there exists $T > 0$ such that for all $u > T$

$$|\delta(u)| \le \varepsilon, \tag{21}$$

where we denoted

$$\delta(u) := M(u) - \frac{u}{\mu} - \frac{\sigma^2-\mu^2}{2\mu^2}. \tag{22}$$

Use the change of variable $u = t - s$ in the integral

$$\int_0^t M(t-s)\beta e^{-\beta s} ds = \int_0^t M(u)\beta e^{-\beta(t-u)} du \tag{23}$$

and split the interval of integration

$$\int_0^t M(u)\beta e^{-\beta(t-u)} du = \int_0^T M(u)\beta e^{-\beta(t-u)} du + \int_T^t M(u)\beta e^{-\beta(t-u)} du. \tag{24}$$

The first term vanishes as $t \to \infty$ (remember that $M(t)$ is nondecreasing and bounded for any $t > 0$)

$$\lim_{t\to\infty} \int_0^T M(u)\beta e^{-\beta(t-u)} du = \lim_{t\to\infty} e^{-\beta t} \int_0^T M(u)\beta e^{\beta u} du = 0. \tag{25}$$

For the second term use (22)

$$\int_T^t M(u)\beta e^{-\beta(t-u)}du = \int_T^t \Big(\frac{u}{\mu} + \frac{\sigma^2-\mu^2}{2\mu^2}\Big)\beta e^{-\beta(t-u)}du + \int_T^t \delta(u)\beta e^{-\beta(t-u)}du. \tag{26}$$

The last term in the above equation is bounded by ε

$$\Big|\int_T^t \delta(u)\beta e^{-\beta(t-u)}du\Big| \le \varepsilon \int_T^t \beta e^{-\beta(t-u)}du \le \varepsilon. \tag{27}$$

It remains to evaluate

$$\lim_{t\to\infty}\int_T^t \Big(\frac{u}{\mu} + \frac{\sigma^2-\mu^2}{2\mu^2}\Big)\beta e^{-\beta(t-u)}du. \tag{28}$$

Change the variable back to $s = t-u$ and split into three integrals

$$\int_0^{t-T} \Big(\frac{t-s}{\mu} + \frac{\sigma^2-\mu^2}{2\mu^2}\Big)\beta e^{-\beta s}ds \tag{29}$$

$$= \frac{t}{\mu}\int_0^{t-T} \beta e^{-\beta s}ds - \frac{1}{\mu}\int_0^{t-T} s\beta e^{-\beta s}ds + \int_0^{t-T} \frac{\sigma^2-\mu^2}{2\mu^2}\beta e^{-\beta s}ds \tag{30}$$

Direct computation gives

$$\lim_{t\to\infty}\int_0^{t-T} \frac{\sigma^2-\mu^2}{2\mu^2}\beta e^{-\beta s}ds = \frac{\sigma^2-\mu^2}{2\mu^2} \tag{31}$$

and

$$\lim_{t\to\infty}\frac{1}{\mu}\int_0^{t-T} s\beta e^{-\beta s}ds = \frac{1}{\mu}\int_0^{\infty} s\beta e^{-\beta s}ds = \frac{1}{\beta\mu}. \tag{32}$$

In the term that remains we identify the leading term $\frac{t}{\mu}$, and a vanishing term

$$\frac{t}{\mu}\int_0^{t-T} \beta e^{-\beta s}ds = \frac{t}{\mu} + \frac{t}{\mu}\int_{t-T}^{\infty} \beta e^{-\beta s}ds = \frac{t}{\mu} + \frac{t}{\mu}e^{-\beta(t-T)} \tag{33}$$

with

$$\lim_{t\to\infty}\frac{t}{\mu}e^{-\beta(t-T)} = 0. \tag{34}$$

Combining (20) - (34) gives

$$\lim_{t\to\infty}\Big(E(N(t)) - \frac{t}{\mu}\Big) = 1 + \frac{\sigma^2-\mu^2}{2\mu^2} - \frac{1}{\beta\mu} + \lim_{t\to\infty}\int_T^t \delta(u)\beta e^{-\beta(t-u)}du \tag{35}$$

$$= 1 - \frac{\alpha\beta}{(\alpha+\beta)^2} - \frac{\alpha}{\alpha+\beta} + \lim_{t\to\infty}\int_T^t \delta(u)\beta e^{-\beta(t-u)}du \tag{36}$$

$$= \frac{\beta}{\alpha+\beta} - \frac{\alpha\beta}{(\alpha+\beta)^2} + \lim_{t\to\infty}\int_T^t \delta(u)\beta e^{-\beta(t-u)}du. \tag{37}$$

Thus

$$\lim_{t\to\infty}\Big|E(N(t)) - \frac{t}{\mu} - \frac{\beta}{\alpha+\beta} + \frac{\alpha\beta}{(\alpha+\beta)^2}\Big| = \lim_{t\to\infty}\Big|\int_T^t \delta(u)\beta e^{-\beta(t-u)}du\Big| \le \varepsilon. \tag{38}$$

Since $\varepsilon > 0$ was an arbitrary constant, we conclude that

$$\lim_{t\to\infty}\left(E(N(t)) - \frac{t}{\mu}\right) = \frac{\beta}{\alpha+\beta} - \frac{\alpha\beta}{(\alpha+\beta)^2}. \tag{39}$$