MATH 180C HOMEWORK 2

SPRING 2023

Due date: **Friday 4/21/2023 11:59 PM** (via Gradescope)
Note that there are *Exercises* and *Problems* in the textbook. Make sure you read the homework carefully to find the assigned question.

1. *Pinsky and Karlin, Exercise 6.2.4.*
    Consider the linear death process in which $X_0 = N = 5$ and $\alpha = 2$. Determine $P(X_t = 2)$.

2. *Pinsky and Karlin, Exercise 6.3.2.*
    Patients arrive at a hospital emergency room according to a Poisson process of rate λ. The patients are treated by a single doctor on a first come, first served basis. The doctor treats patients more quickly when the number of patients waiting is higher. An industrial engineering time study suggests that the mean patient treatment time when there are k patients in the system is of the form $m_k = \alpha - \beta \frac{k}{k+1}$, where α and β are constants with $\alpha > \beta > 0$. Let $N(t)$ be the number of patients in the system at time t (waiting and being treated). Argue that $N(t)$ might be modeled as a birth and death process with parameters $\lambda_k = \lambda$ for $k = 0, 1, \ldots$ and $\mu_k = \frac{1}{m_k}$ for $k = 0, 1, \ldots$. State explicitly any necessary assumptions.

3. *Matrix exponentials*
    Let

(1)
$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

   Compute e^{tC}. [Hint. It might be helpful to check the Taylor (Maclaurin) series expansions for $\sin(t)$ and $\cos(t)$.]
   Use this result to show that for matrices A, B the following equality is *not* generally true

(2)
$$e^{(A+B)} = e^A e^B.$$

4. *Pinsky and Karlin, Problem 6.3.1.*
    Let $\xi_n, n = 0, 1, \ldots$, be a two-state Markov chain with transition probability matrix

(3)
$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & \quad 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \left\| \begin{matrix} 0 & 1 \\ 1-\alpha & \alpha \end{matrix} \right\|. \end{matrix}$$

Let $\{N(t); t \geq 0\}$ be a Poisson process with parameter λ independent of $(\xi_n)_{n \geq 0}$. Show that

(4) $$X_t = \xi_{N(t)}, \quad t \geq 0,$$

is a two-state birth and death process and determine the parameters λ_0 and μ_1 in terms of α and λ.

5. *Pinsky and Karlin, Exercise 6.6.2.*
   Let $X_1(t)$ and $X_2(t)$ be independent two-state Markov chains having the same infinitesimal matrix

(5) $$\mathbf{A} = \begin{matrix} & \begin{matrix} 0 & \quad 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \left\| \begin{matrix} -\lambda & \lambda \\ \mu & -\mu \end{matrix} \right\| \end{matrix}.$$

Argue that $Z(t) = X_1(t) + X_2(t)$ is a Markov chain on the state space $S = \{0, 1, 2\}$ and determine the transition probability matrix $\mathbf{P}(t)$ for $Z(t)$.