Name (last, first): ______________________________

Student ID: ______________________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **The exam consists of 8 questions. Your answers must be carefully justified to receive credit.**

☐ **This exam will be scanned. Make sure you write ALL SOLUTIONS on the paper provided. DO NOT REMOVE ANY OF THE PAGES.**

☐ **No calculators, phones, or other electronic devices are allowed.**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You are allowed to use two 8.5 by 11 inch sheets of paper with hand-written notes (on both sides); no other notes (or books) are allowed.**

1. (10 points) There are two urns. The first urn has 3 red balls, 2 blue balls and 2 green balls. The second urn has 2 red balls and 4 blue balls. You choose one of the urns at random (with equal probability), and then sample one ball from that urn. The ball that you picked is blue.

   What is the probability that the ball was picked from the first urn?

   Solution. Denote by B_i the event that the ith urn has been chosen, and by A the event that you have picked a blue ball. Then

$$P(B_1) = P(B_2) = \frac{1}{2}, \quad P(A|B_1) = \frac{2}{7}, \quad P(A|B_2) = \frac{4}{6}, \tag{1}$$

   and using the Bayes' rule we have that

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2)} = \frac{\frac{2}{7}\cdot\frac{1}{2}}{\frac{2}{7}\cdot\frac{1}{2} + \frac{4}{6}\cdot\frac{1}{2}} = \frac{3}{10}. \tag{2}$$

2. (10 points) Suppose that you roll a fair 6-sided die five times.

(a) What is the probability that exactly three of your rolls are even numbers?

Solution. We describe this experiment using the model of sampling with replacement (order matters). Then

$$\Omega = \{1,2,3,4,5,6\}^5, \qquad \#\Omega = 6^5. \tag{3}$$

Denote

$$A = \{\text{exactly 3 are even numbers}\}. \tag{4}$$

Then

$$\#A = \binom{5}{3} 3^3\, 3^2, \tag{5}$$

where the first factor gives the number of different positions for 3 even numbers among five rolls, the second factor gives the number of different triples of even numbers between 1 and 6, and the last term gives the number of possible pairs of odd numbers between 1 and 6. Then

$$P(A) = \frac{10 \cdot 3^5}{6^5} = \frac{10}{2^5} = \frac{5}{16}. \tag{6}$$

(b) What is the probability that your first roll and your last roll are equal?

Solution. Denote

$$B = \{\text{first and last rolls are equal}\}. \tag{7}$$

Then

$$\#B = 6 \cdot 6^3, \tag{8}$$

where the first factor is the number of possible choices for the first and the last rolls, and the second factor is the number of possible choices for the other three rolls. Then

$$P(B) = \frac{6^4}{6^5} = \frac{1}{6}. \tag{9}$$

(c) What is the probability that no two of your five rolls are equal?

Solution. Denote

$$C = \{\text{all rolls are different}\}. \tag{10}$$

Then

$$\#C = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 6!, \tag{11}$$

and

$$P(C) = \frac{6!}{6^5}. \tag{12}$$

3. (10 points) According to the US Department of Treasury, one in every 10,000 US dollar notes is counterfeit.

   A cash-in-transit van operating in San Diego area transports 20,000 US dollar notes from a supermarket to a bank.

   Estimate the probability that there are at least 3 counterfeit notes in this van.

   [Explain your choice of approximation. You may leave the answers in terms of $\Phi(x)$ or e^x. Do not use the continuity correction.]

   Solution. We model the number of the counterfeit notes in the van using a random variable X having binomial distribution with parameters $n = 20000$ and $p = 10^{-4}$, $X \sim \text{Bin}(20000, 10^{-4})$. We are asked to compute the probability that $X \geq 3$. For this, we use the Poisson approximation with $\lambda = np = 20000 \cdot 10^{-4} = 2$, so that for any $k \in \{0, 1, 2, \ldots\}$

$$P(X = k) \approx \frac{2^k}{k!} e^{-2}. \tag{13}$$

   We choose the Poisson approximation since $np^2 = 2 \cdot 10^{-4}$ is much smaller than 1, and

$$np(1-p) = 20000 \cdot 10^{-4}(1 - 10^{-4}) \approx 2, \tag{14}$$

   which is not enough to guarantee a good normal approximation.

   In order to estimate $P(X \geq 3)$ we use the complement formula together with the Poisson approximation

$$P(X \geq 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)] \approx 1 - \left[e^{-2} + \frac{2}{1!}e^{-2} + \frac{4}{2!}e^{-2}\right] = 1 - 5e^{-2}. \tag{15}$$

4. (10 points) Let X be a random variable describing the lifetime of a certain component. We know that X is a continuous random variable with PDF (in years)

$$f_X(x) = \begin{cases} \frac{2}{x^3}, & x \in [1, +\infty), \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

The component is replaced either when it fails, or after 4 years of service even if it is still operational.

Denote by Y the time the component is in operation (i.e., from being installed to being replaced). Compute $E(Y)$.

Solution. The time in operation Y is equal to X if X is less than 4 years, or equal to 4 if X is greater than 4 years. Therefore, we can express Y as a function of X

$$Y = g(X), \tag{17}$$

where

$$g(x) = \begin{cases} x, & x \leq 4, \\ 4, & x > 4. \end{cases} \tag{18}$$

We can thus compute $E(Y)$ as $E(g(X)]$

$$E(Y) = \int_1^{+\infty} g(x) f_X(x) dx = \int_1^4 x \cdot \frac{2}{x^3} dx + \int_4^{+\infty} 4 \cdot \frac{2}{x^3} dx \tag{19}$$

The first integral gives

$$\int_1^4 \frac{2}{x^2} dx = -\frac{2}{x}\Big|_1^4 = 2 - \frac{1}{2} = \frac{3}{2}, \tag{20}$$

the second integral gives

$$\int_4^{+\infty} \frac{8}{x^3} dx = -\frac{8}{2x^2}\Big|_4^{\infty} = \frac{1}{4}. \tag{21}$$

Therefore,

$$E(Y) = \frac{3}{2} + \frac{1}{4} = \frac{7}{4}. \tag{22}$$

5. (10 points) Let X be a continuous random variable with density

$$f_X(x) = \begin{cases} xe^{-x}, & x > 0. \\ 0, & x \le 0. \end{cases}$$

(a) Compute the moment generating function $M_X(t)$. Indicate for which values $t \in \mathbb{R}$ function $M_X(t)$ is finite.

Solution. We compute the moment generating function using the definition $M_X(t) = E(e^{tX})$

$$M_X(t) = \int_0^\infty e^{tx} x e^{-x} dx = \int_0^\infty x e^{(t-1)x} dx. \tag{23}$$

This integral diverges if $t \ge 1$, and for $t < 1$ using the integration by parts we get

$$M_X(t) = \frac{1}{t-1}\left(xe^{(t-1)x}\Big|_0^\infty - \int_0^\infty e^{(t-1)x} dx \right) = \frac{1}{(1-t)^2}. \tag{24}$$

So

$$M_X(t) = \begin{cases} \frac{1}{(1-t)^2}, & t < 1, \\ \infty, & \text{otherwise.} \end{cases} \tag{25}$$

(b) Compute the first three moments of X, i.e., $E(X)$, $E(X^2)$, and $E(X^3)$, using $M_X(t)$.

Solution. Differentiating $M_X(t)$ repeatedly for $t < 1$ gives

$$M_X'(t) = \frac{2}{(1-t)^3}, \quad M_X''(t) = \frac{2 \cdot 3}{(1-t)^4}, \quad M_X'''(t) = \frac{2 \cdot 3 \cdot 4}{(1-t)^5}. \tag{26}$$

Thus

$$E(X) = M_X'(0) = 2, \quad E(X^2) = M_X''(0) = 6, \quad E(X^3) = M_X'''(0) = 24. \tag{27}$$

(c) Extra 5 points: Compute $E(X^n)$ for general $n \ge 1$ using $M_X(t)$. [Hint: use

$$\sum_{k=0}^\infty a_k t^k \cdot \sum_{\ell=0}^\infty b_\ell t^\ell = \sum_{k=0}^\infty \sum_{\ell=0}^\infty a_k \, b_\ell \, t^{k+\ell}$$

and the geometric series formula]

Solution. To compute the moments of X, write the Taylor series of $M_X(t)$ at $t = 0$ using the geometric series expansion $(1-t)^{-1} = \sum_{k=0}^\infty t^k$

$$M_X(t) = \frac{1}{1-t} \cdot \frac{1}{1-t} = \sum_{k=0}^\infty t^k \sum_{\ell=0}^\infty t^\ell = \sum_{k=0}^\infty \sum_{\ell=0}^\infty t^{k+\ell}. \tag{28}$$

By introducing the index $m = k + \ell$, (28) can be rewritten as

$$M_X(t) = \sum_{k=0}^\infty \sum_{\ell=0}^\infty t^{k+\ell} = \sum_{m=0}^\infty (m+1) t^m = \sum_{m=0}^\infty (m+1)! \frac{t^m}{m!}, \tag{29}$$

and we conclude that

$$E(X^m) = (m+1)!\,. \tag{30}$$

6. (10 points) Let X and Y be a pair of jointly continuous random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} c(x+3y), & 0 \le x, y \le 1, \\ 0, & \text{otherwise}, \end{cases} \tag{31}$$

with an unknown parameter $c > 0$.

(a) Determine the value of $c > 0$.

Solution. From

$$\int_0^1 \int_0^1 f_{X,Y}(x,y)dxdy = c\int_0^1 \int_0^1 (x+3y)dxdy = c\Big(\frac{1}{2} + \frac{3}{2}\Big) = 2c = 1 \tag{32}$$

we find that $c = 1/2$.

(b) Compute the marginal densities of X and Y.

Solution.

$$f_X(x) = \int_0^1 \frac{1}{2}(x+3y)dy = \frac{x}{2} + \frac{3}{4}, \quad f_Y(y) = \int_0^1 \frac{1}{2}(x+3y)dx = \frac{1}{4} + \frac{3y}{2}. \tag{33}$$

(c) Determine if random variables X and Y are independent.

Solution. Since $f_X(x)f_Y(y) \neq f_{X,Y}(x,y)$, random variables X and Y are not independent.

7. (10 points) Suppose that we choose a number N uniformly at random from the set $\{0, \dots, 4999\}$. Let X denote the sum of its digits. For example, if $N = 123$, then $X = 1+2+3 = 6$. Determine $E(X)$.

Solution.

Note that $X = Y_1 + Y_2 + Y_3 + Y_4$, where Y_i is the ith digit of the number that is drawn. In other words,

$$N = 1000Y_1 + 100Y_2 + 10Y_3 + Y_4.$$

Next, we see that

$$Y_1 \sim \text{Unif}\{0, 1, \dots 4\}, \qquad Y_2, Y_3, Y_4 \sim \text{Unif}\{0, 1, \dots, 9\}.$$

This means that

$$E(Y_1) = 2, \qquad E(Y_2) = E(Y_2) = E(Y_3) = 4.5.$$

So,

$$E(X) = \sum_{i=1}^{4} E(Y_i) = 2 + 3(4.5) = 15.5.$$

8. (10 points) Let X and Y be independent random variables with $X \sim \mathcal{N}(1,4)$ and $Y \sim \mathrm{Exp}(1)$. [**Hint**. This problem can be solved without writing or computing any integrals.]

   (a) Compute $E(2X-3Y+1)$ and $\mathrm{Var}(2X-3Y+1)$.

   Solution. We start by noticing that

$$E(X)=1, \quad \mathrm{Var}(X)=4, \quad E(Y)=1, \quad \mathrm{Var}(Y)=1. \tag{34}$$

   By the linearity of expectation

$$E(2X-3Y+1)=2E(X)-3E(Y)+1=0. \tag{35}$$

   If we additionally use the independence of X and Y, we also have

$$\mathrm{Var}(2X-3Y+1)=4\mathrm{Var}(X)+9\mathrm{Var}(Y)=25. \tag{36}$$

   (b) Compute $\mathrm{Var}(XY)$.

   Solution. We start by writing

$$\begin{aligned}
\mathrm{Var}(XY) &= E\big((XY)^2\big)-\big(E(XY)\big)^2 && (37)\\
&= E\big(X^2Y^2\big)-\big(E(XY)\big)^2 && (38)\\
&= E\big(X^2\big)E\big(Y^2\big)-\big(E(X)\big)^2\big(E(Y)\big)^2, && (39)
\end{aligned}$$

   where on the last step we used the independence of X and Y. We now compute $E(X^2)$ and $E(Y^2)$

$$\mathrm{Var}(X)=E(X^2)-\big(E(X)\big)^2=E(X^2)-1=4 \quad \Rightarrow \quad E(X^2)=5, \tag{40}$$

$$\mathrm{Var}(Y)=E(Y^2)-\big(E(Y)\big)^2=E(Y^2)-1=1 \quad \Rightarrow \quad E(Y^2)=2. \tag{41}$$

   Now we substitute $E(X^2)=5$ and $E(Y^2)=2$ into (39) to get the final answer

$$\mathrm{Var}(XY)=5\cdot 2-1\cdot 1=9. \tag{42}$$