MATH 180A (Lecture A00)

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Today: Random variables

Next: ASV 3.2

Week 3:

- homework 2 (due Sunday, January 29)
- Midterm 1 (Wednesday, February 1, lectures 1-8)
- 5 homework extension days per student per quarter

Independence for more than two events

<u>Def</u>. A collection of events $A_1, A_2, \dots, A_n$ is **mutually independent** if for any subcollection of events $A_{i_1}, A_{i_2}, \dots, A_{i_k}$ with $1 \le i_1 < i_2 < \dots < i_k \le n$

$$P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = P(A_{i_1}) P(A_{i_2}) \cdots P(A_{i_k})$$

<u>Example</u> For $n=3$, A, B, C are mutually independent if

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cap C) = P(A)P(C)$$

$$P(B \cap C) = P(B)P(C)$$

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

Suppose that A and B are independent, A and C are independent, B and C are independent. Are A, B, C mutually independent?

Important example

Toss a coin

$A = \{$ there is exactly one tails in the first two tosses $\}$

$B = \{$ there is exactly one tails in the last two tosses $\}$

$C = \{$ there is exactly one tails in the first and last tosses $\}$

$A = \{(H, T, *), (T, H, *)\}$ $B = \{(*, H, T), (*, T, H)\}$

$C = \{(H, *, T), (T, *, H)\}$ $A \cap B = \{THT, HTH\}$

$P(A) = \frac{4}{8} = \frac{1}{2} = P(B) = P(C)$

$P(A \cap B) = P(A)P(B)$

$P(A \cap B) = \frac{2}{8} = \frac{1}{4} = P(B \cap C) = P(A \cap C)$, A, B, C are pairwise independent

$P(A \cap B \cap C) = 0 \neq P(A)P(B)P(C)$ $A \cap B \cap C = \emptyset$

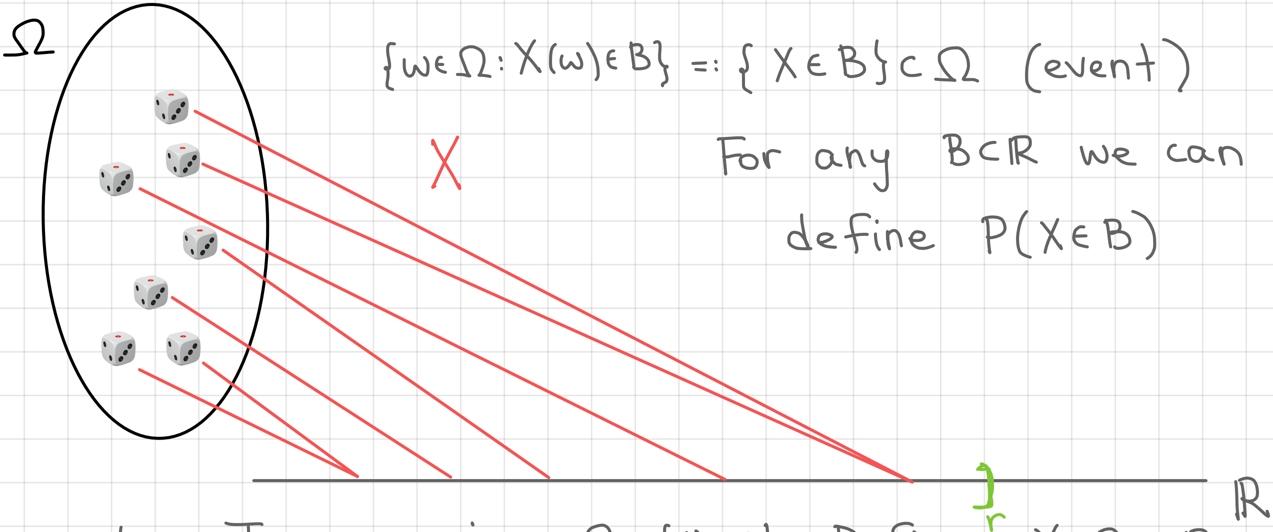
Random variables

$(\Omega, \mathcal{F}, P)$ - probability space

Def A (measurable^{+}) function $X : \Omega \to \mathbb{R}$ is called a random variable.

$\{\omega \in \Omega : X(\omega) \in B\} =: \{X \in B\} \subset \Omega$ (event)

For any $B \subset \mathbb{R}$ we can define $P(X \in B)$



$X = \#$ heads

Example Toss a coin. $\Omega = \{H, T\}$. Define $X : \Omega \to \mathbb{R}$

$X(H) = 1$, $X(T) = 0$ $\quad P(X \in \{1\}) = P(X = 1) = P(\{H\})$, $P(X=0) = P(T$

Probability distribution

Def Let X be a random variable. The **probability distribution** of X is the collection of probabilities

$$P(X \in B) \quad \text{for all } B \subset \mathbb{R}$$

Remark

Examples 1) Coin toss: $\Omega = \{H, T\}$, $X(H) = 1$, $X(T) = 0$

$$P(X=0) = P(\{T\}) = \frac{1}{2} = P(X=1) \quad \text{(fair coin)}$$

2) Roll a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\quad X(\omega) = \omega$

For any $1 \le i \le 6$, $\quad P(X = i) = \frac{1}{6}$

Probability distribution

3) Roll a die twice : $\Omega = \{(i,j) : i,j \in \{1,2,\dots,6\}\}$

$X_1((i,j)) = i$ (first number) , $X_2((i,j)) = j$ second number

for $1 \le i \le 6$ $P(X_1 = i) = \frac{1}{6}$, $P(X_2 = i) = \frac{1}{6}$

Define $S = X_1 + X_2$

$P(S=2) = \frac{1}{36}$ $\quad$ $P(S=7) = \frac{6}{36}$

$P(S=3) = \frac{2}{36}$ $\quad$ $P(S=8) = \frac{5}{36}$

$P(S=4) = \frac{3}{36}$ $\quad$ $P(S=9) = \frac{4}{36}$

$P(S=5) = \frac{4}{36}$ $\quad$ $P(S=10) = \frac{3}{36}$

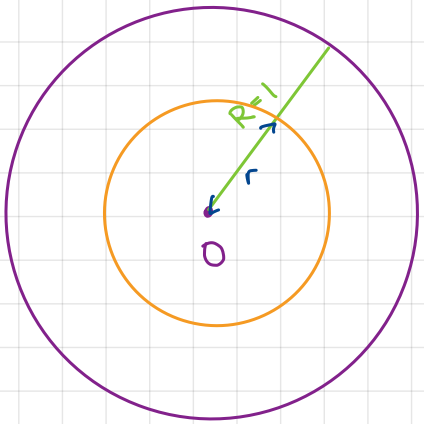
$P(S=6) = \frac{5}{36}$ $\quad$ $P(S=11) = \frac{2}{36}$

$\parallel$ $\quad$ $P(S=12) = \frac{1}{36}$

$\mu_X(\{6\})$

Probability distribution

4) Choosing a point from unit disk uniformly at random



$\Omega = \{\omega \in \mathbb{R}^2 : dist(0,\omega) \le 1\}$

$X(\omega) = dist(0, \omega)$

For any $r < 0$, $P(X \le r) = 0$

For any $r > 1$, $P(X \le r) = 1$

For any $0 \le r \le 1$, $P(X \le r) = \frac{\pi r^2}{\pi} = r^2$

Probability distribution

If $(\Omega, \mathcal{F}, P)$ is a probability space, and $X : \Omega \to \mathbb{R}$ is a random variable, we can define a probability measure μ_X on $\mathbb{R}$ given, for any $A \subset \mathbb{R}$, by

$$\mu_X(A) = P(X \in A) = P(\{\omega : X(\omega) \in A\})$$

We call μ_X the probability distribution (or law) of X.

5) Toss a fair coin 4 times. Let X = number of tails

$$\Omega = \{(X_1, X_2, X_3, X_4) \in \{H, T\}^4\}$$

P = uniform on Ω

$$P((X_1, X_2, X_3, X_4)) = \frac{1}{2^4} = \frac{1}{16}$$

$$\mu_X(\{2\}) = P(X = 2) = \frac{\binom{4}{2}}{16} = \binom{4}{2}\frac{1}{16}$$

$$\mu_X(\{k\}) = \binom{4}{k} \cdot \frac{1}{16}$$

$$X \in \{0, 1, 2, 3, 4\}$$

If $A \subset \mathbb{R}$ does not contain one of these numbers, then

$$\mu_X(A) = 0$$

Enough to know $\mu_X(\{k\})$ for $0 \leq k \leq 4$