MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

Today: Consequences of the axioms of probability

Next: ASV 1.3-1.4

Week 2:

- homework 1 (due ~~Friday, January 20~~)
  Monday, Jan 23

Infinite sample space

If $\#\Omega = \infty$, then we need a different notion of uniform probability measure.

Example A random number is chosen in $[0,1]$.

(a) What is the probability that it is ≥ 0.6

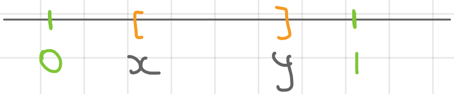
(b) What is the probability that it is $= \frac{1}{2}$

$(\Omega, \mathcal{F}, P)$: $\Omega = [0,1]$

$$P([x,y]) = y-x$$

(a) $P([0.6,1]) = 1-0.6 = 0.4$

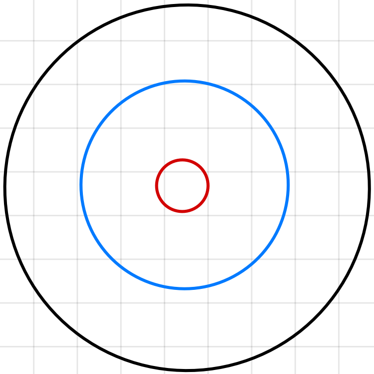
(b) $P([\frac{1}{2},\frac{1}{2}]) = \frac{1}{2} - \frac{1}{2} = 0$



$$y-x = P([x,y]) = P([x,y) \cup \{y\})$$
$$= P([x,y)) + P(\{y\})$$

If $\Omega = [a,b]$, then take $P([x,y]) = \frac{y-x}{b-a}$ for $[x,y] \subset [a,b]$

Infinite sample space



An archery target is a disk 50 cm in diameter

A blue disk is 25 cm in diameter

A red disk is 5 cm in diameter

Given that you hit the target (randomly), what are the chances of hitting the blue disk? The red disk?

Ω = target, $P(A) = \dfrac{\text{Area}(A)}{\text{Area}(\Omega)}$, $\mathcal{F} = \{\text{subsets with area}\}$

$$P(\text{blue disk}) = \frac{\pi\left(\frac{5}{2}\right)^2}{\pi\left(\frac{50}{2}\right)^2} = \frac{1}{100}$$

General rule: uniform probability $P(A) = \dfrac{\text{"size"}(A)}{\text{"size"}(\Omega)}$

Decompositions

Example A fair die is rolled 5 times. What is the probability that you get one at least 3 times?

$$A = \{\text{at least } 3 \text{ ones}\} = A_3 \cup A_4 \cup A_5$$

$$A_k := \{\text{exactly } k \text{ ones}\}$$

$$P(A) = P(A_3) + P(A_4) + P(A_5)$$

Consider A_3, exactly 3 ones. How many ways?

$\square \quad \square \quad \square \quad \square \quad \square$

1) # of configurations (patterns) $= \binom{5}{3}$

2) # of ways for each configuration $= 5^2$

$$P(A_3) = \frac{\binom{5}{3} \cdot 5^2}{6^5}, \quad P(A_4) = \frac{\binom{5}{4} \cdot 5}{6^5}, \quad P(A_5) = \frac{\binom{5}{5} \cdot 1}{6^5}$$

$$P(A) = \frac{1}{6^5}\left(10 \cdot 25 + 5 \cdot 5 + 1\right) = \frac{276}{7776} \approx 3.55\,\%$$

Decompositions

Example A fair die is rolled 5 times. What is the probability that you get at least one double?

$A = \{$some number comes up at least twice$\}$

$A_k = \{k$ comes up at least two times$\}$

$$A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$$

We can further decompose

$A_1 = A_1^2 \cup A_1^3 \cup A_1^4 \cup A_1^5 \cup A_1^6$, $A_1^j = \{1$ comes up exactly j times$\}$

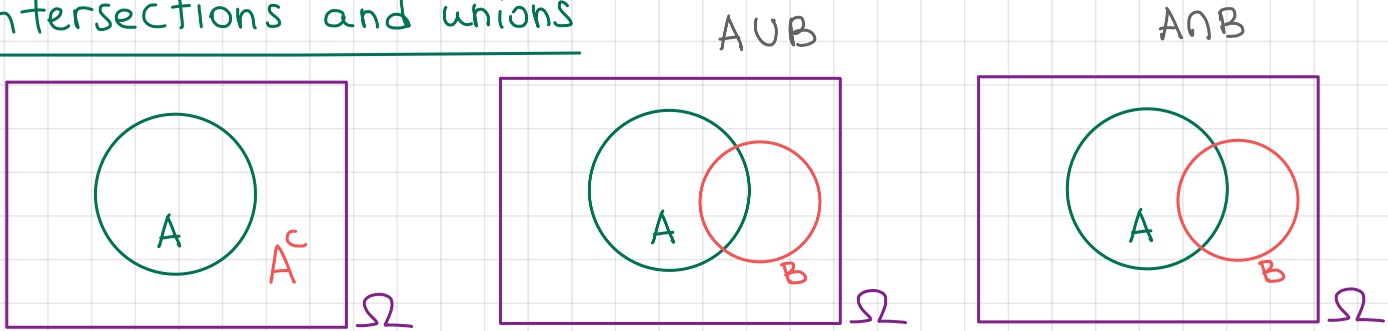
$\vdots$

$A_1, A_2, \dots, A_6$ are not disjoint! Avoid overcounting!

Easy solution: consider the complement $A^c = \{$no doubles$\}$

$\# A^c = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 6!$, $P(A^c) = \frac{6!}{6^5}$, $P(A \cup A^c) = P(A) + P(A^c) = 1$

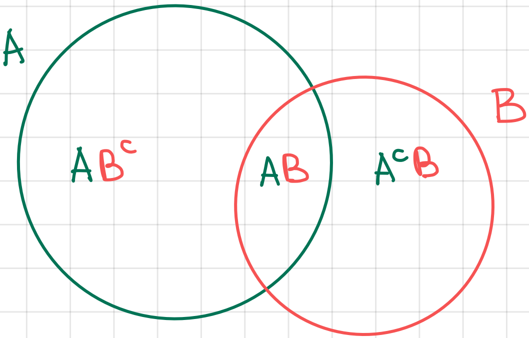
Intersections and unions



Sometimes we have to take intersections into account.

Notation: $A \cup B = \{\text{all outcomes in either } A \text{ or } B \text{ or both}\}$

$A \cap B = \{\text{all outcomes in both } A \text{ and } B\} = AB$

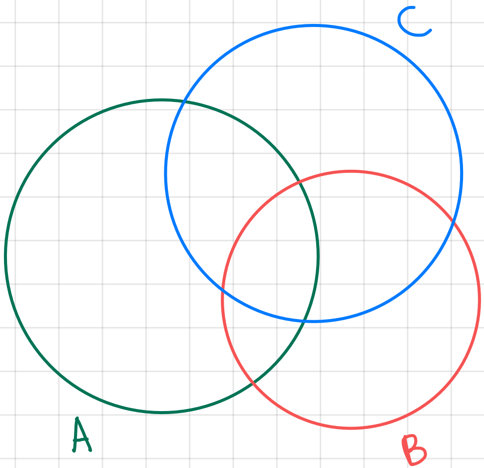


$$A \cup B = \underbrace{AB^c \cup AB \cup A^cB}_{\text{disjoint}}$$

$$P(A \cup B) = P(A) + P(B) - P(AB)$$

Principle of inclusion / exclusion

The probability of a union can be computed by adding the probabilities, then subtracting off the intersection overcounted. If you have more sets, you have to keep going and re-add back in over-subtracted pieces etc...



$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

$$P(A\cup B\cup C\cup B) = P(A) + P(B) + P(C) + P(D) - P(AB) - P(AC) - \cdots + P(ABC) + P(ABD) - \cdots - P(ABCD)$$