MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

Today: Definition of probability. Random sampling

Next: ASV 2.1

Week 1:

- check the course website
- homework 1 (due Friday, January 20)
- join Piazza

Last time

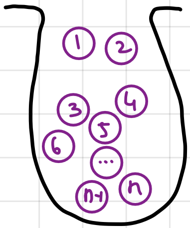
If Ω is finite, the uniform probability measure is defined by the following property:

$$\text{for each } \omega \in \Omega, \quad P(\{\omega\}) = \frac{1}{\#\Omega}$$

From (*) this implies that

$$\text{for any event } A, \quad P(A) = \frac{\#A}{\#\Omega}$$

Combinatorics



A collection of n labelled balls $\{1, 2, 3, \dots, n\}$ are in an urn. k are taken out one by one.

Q: How many ways?

Possible scenarios:

Replacement
- with replacement
- without replacement

Order
- order balls come out matters
- order does not matter

$n = 5$, $k = 3$ (choose 3 balls)	order matters	order doesn't matter
with replacement	1 2 1 $\neq$ 1 1 2 (b_1, b_2, b_3)	1 2 1 $=$ 1 1 2 1 1 1
without replacement	1 2 3 $\neq$ 3 2 1 (b_1, b_2, b_3), $b_i \neq b_j$ if $i \neq j$	1 2 3 $=$ 3 2 1 $\{b_1, b_2, b_3\}$

Combinatorics

Sampling with replacement, order matters

$$\Omega = \{(b_1,\dots,b_k) : 1 \le b_i \le n\} = \{1,\dots,n\}^k$$

Sampling without replacement, order matters

$$\Omega = \{(b_1,\dots,b_k) : 1 \le b_j \le n,\ b_i \ne b_j \text{ if } i \ne j\}$$

Sampling without replacement, order does not matter

$$\Omega = \{\{b_1,\dots,b_k\} : 1 \le b_i \le n,\ b_i \ne b_j \text{ if } i \ne j\}$$

	order matters	order doesn't matter
with replacement	$\#\Omega = \underbrace{n \cdot n \cdots n}_{k} = n^k$	
without replacement	$\#\Omega = n\cdot(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$	$\#\Omega = \frac{n\cdot(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$

if $k=n$, $\#\Omega = n!$ ← # ways to order n elements

Important remark. Examples

Each of these three models leads to a
uniform probability measure!
on the corresponding sample space

Example (sampling with replacement)

Toss a fair coin n times; record a statistic observing
$\# H$ vs $\# T$

Take $n = 10$. Q: compute P(odd rolls are all H)

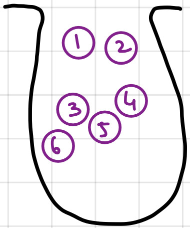
$$\Omega = \{ (c_1, c_2, \dots, c_{10}) : c_j \in \{H, T\} \}, \quad \#\Omega = 2^{10}$$

$$A = \{ c_1 = c_3 = c_5 = c_7 = c_9 = H \} = \{ (H, *, H, *, H, *, H, *, H, *) \}$$

$$\# A = 2^5, \quad P(A) = \frac{\# A}{\# \Omega} = \frac{2^5}{2^{10}} = \frac{1}{2^5} = \frac{1}{32}$$

Examples

Example (Sampling without replacement, order matters)



There are 6 labelled balls in an urn.
3 are removed in sequence (without replacement) and lined up in order.

Q: What is the probability that the first two are (4, 5)?

$$\Omega = \{ (b_1, b_2, b_3) : 1 \le b_j \le 6, \ b_1 \ne b_2, \ b_2 \ne b_3, \ b_1 \ne b_3 \}$$

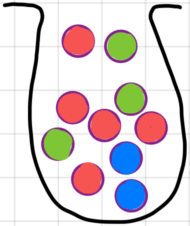
$$\# \Omega = 6 \cdot 5 \cdot 4 = 120$$

$$A = \{ (4, 5, b_3), \ 1 \le b_3 \le 6, \ b_3 \ne 4, \ b_3 \ne 5 \}, \quad \# A = 4$$

$$P(A) = \frac{4}{120} = \frac{1}{30}$$

Examples

Example (Sampling without replacement, order does not matter)



An urn contains 10 balls: $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}$

two blue, three green, five red

3 balls are chosen without replacement.

Q: Compute P(choose 2 green and one red)

$$\#\Omega = \binom{10}{3} = \frac{10!}{3!\,7!} = \frac{8\cdot 9\cdot 10}{2\cdot 3} = 120$$

\# ways to choose 2 green out of 3

$$A = \{\text{2 are green, 1 red}\} = \binom{3}{2}\binom{5}{1} = 3\cdot 5 = 15$$

$$P(A) = \frac{\#A}{\#\Omega} = \frac{15}{120} = \frac{1}{8}$$

Combinatorics

- selecting k objects among n, with replacement

$$\# \text{ways} = n^k$$

- selecting k objects among n, without replacement
  order matters

$$\# \text{ways} = n(n-1)(n-2)\cdots(n-k+1) \qquad (k \leq n)$$

- selecting k objects among n, without replacement
  order doesn't matter

$$\# \text{ways} = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

- # of ways to order n objects : $n(n-1)\cdots 1 = n!$

Warm-up exercise

There are 10 people on a committee.

How many different ways are there to select a subcommittee of 4 people?

(a) $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10\,000$

(b) $10 \cdot 9 \cdot 8 \cdot 7 = 5040$

(c) $\binom{10}{4} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7}{4!} = 210$

(d) $\dfrac{10!}{4!} = 151\,200$

Example

You have a deck of 52 cards (4 suits × 13 ranks).
You choose 5 cards uniformly at random.
What is the probability that you choose 3 cards of one rank + 2 cards of another rank (full house)?

Ω = sets (unordered) of 5 distinct cards , $\#\Omega = \binom{52}{5}$

A = full house

choose rank of 2 cards

$\{a, b, c\}$

choose suits

$$\#A = \underbrace{13}_{\text{\# choose rank of 3 cards}} \cdot \underbrace{\binom{4}{3}}_{\text{\# choose suits}} \cdot 12 \cdot \binom{4}{2}$$

$$P(\text{full house}) = \frac{\#A}{\#\Omega} = \frac{13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}}{\binom{52}{5}} = \frac{3744}{2\,598\,960} \simeq 0.1441\%$$