MATH 180A (Lecture A00)

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Today: Joint distribution. Independence

Next: ASV 8.1

Week 10:

- Homework 7 due Sunday, March 19

Example

Consider again random variables X, Y with joint density

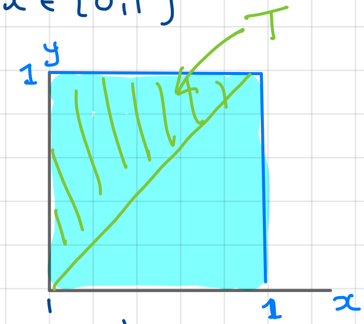
$$f(x,y) = \begin{cases} \frac{3}{2}(xy^2+y) & , \quad 0 \le x \le 1, \ 0 \le y \le 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(i) $E(X^2Y) = \int_0^1 \int_0^1 x^2 y \, \frac{3}{2}(xy^2+y)\, dx\, dy = \frac{3}{2} \int_0^1 \int_0^1 (x^3y^3 + x^2y^2)\, dx\, dy = \cdots = \frac{25}{36}$

$g(x,y) = x^2 y$

(ii) $f_X(x) = \int_0^1 \frac{3}{2}(xy^2+y)\, dy = \frac{3}{2}\left(x \cdot \frac{1}{3} + \frac{1}{2}\right) = \frac{x}{2} + \frac{3}{4}, \quad x \in [0,1]$

$f_Y(y) = \int_0^1 \frac{3}{2}(xy^2+y)\, dx = \frac{3}{4}y^2 + \frac{3}{2}y, \quad y \in [0,1]$



(iii) $P(X<Y) = P((X,Y) \in T) = \iint_T f(x,y)\, dx\, dy = \int_0^1 \left(\int_0^y \frac{3}{2}(xy^2+y)\, dx \right) dy$

$T = \{(x,y) : 0 \le x < y \le 1\}$

$= \int_0^1 \frac{3}{2}\left(\frac{y^4}{2} + y^2\right) dy = \frac{3}{2}\left(\frac{1}{10} + \frac{1}{3}\right)$

Joint distribution and independence

Random variables $X_1, \dots, X_n$ defined on the same probability space are independent if for any $B_1, \dots, B_n \subset \mathbb{R}$

$$P(X_1 \in B_1, X_2 \in B_2, \cdots, X_n \in B_n) = P(X_1 \in B_1) P(X_2 \in B_2) \cdots P(X_n \in B_n)$$

Independence can be expressed in terms of PMF/PDF

Discrete case:

Let $p(k_1, \dots, k_n)$ be the joint PMF of discrete random variables $X_1, \dots, X_n$. Let $p_{X_j}(k) = P(X_j = k)$ be the marginal PMF of X_j. Then $X_1, X_2, \dots, X_n$ are independent if and only if

$$p(k_1, k_2, \dots, k_n) = p_{X_1}(k_1)\, p_{X_2}(k_2) \cdots p_{X_n}(k_n)$$

Examples

Example 1

Roll a fair die twice

$X_1 = \#$ even numbers

$X_2 = \#$ sixes

$K_1 \backslash K_2$	0	1	2	
0	$\frac{9}{36}$	0	0	$\frac{9}{36}$
1	$\frac{12}{36}$	$\frac{6}{36}$	0	$\frac{18}{36}$
2	$\frac{4}{36}$	$\frac{4}{36}$	$\frac{1}{36}$	$\frac{9}{36}$
	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$	

Are X_1 and X_2 independent?

$$p(0,1) = 0 \neq p_{X_1}(0) \cdot p_{X_2}(1)$$

Example 2

joint PMF of X_1, X_2

$K_1 \backslash K_2$	0	1	
0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
1	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$
	$\frac{1}{3}$	$\frac{2}{3}$	

for any k_1, k_2

$$p(k_1, k_2) = p_{X_1}(k_1)\, p_{X_2}(k_2)$$

X_1 and X_2 are independent

Joint distributions and independence. Continuous case

Thm. Let $X_1, \dots, X_n$ be random variables defined on the same probability space. Assume that each X_j has PDF f_{X_j}.

(i) If the joint density of $X_1, \dots, X_n$ is equal to

$$f(x_1, x_2, \dots, x_n) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) \cdot \dots \cdot f_{X_n}(x_n)$$

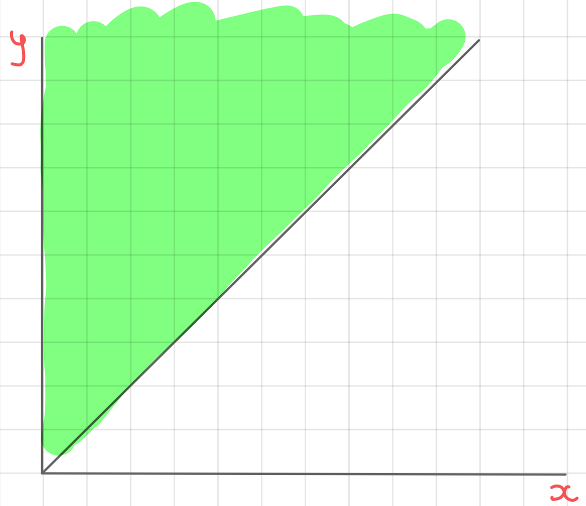
then $X_1, \dots, X_n$ are independent

(ii) If $X_1, \dots, X_n$ are independent then

$$f(x_1, x_2, \dots, x_n) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) \cdot \dots \cdot f_{X_n}(x_n)$$

Thm. Let $X_1, \dots, X_n, X_{n+1}, \dots, X_{n+m}$ be independent random variables. Let $f: \mathbb{R}^n \to \mathbb{R}$, $g: \mathbb{R}^m \to \mathbb{R}$. Define $Y = f(X_1, \dots, X_n)$, $Z = g(X_{n+1}, \dots, X_{n+m})$. Then Y and Z are independent

Joint distribution and independence



Example Let $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$

Suppose X and Y are independent.

1) Find the joint PDF of X and Y

2) Compute $P(X<Y)$

X takes values in $(0,+\infty)$, Y takes values in $(0,+\infty)$

(X,Y) take values in $\{(x,y) : x>0, y>0\}$

1) From the independence $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) = \begin{cases} \lambda e^{-\lambda x} \cdot \mu e^{-\mu y}, & x>0, \ y>0 \\ 0, & \text{otherwise} \end{cases}$

2) $P(X<Y) = P\left((X,Y) \in \{(x,y) : \underset{x>0}{x<y}\}\right) = \int_0^\infty \int_x^\infty \lambda\mu \, e^{-\lambda x - \mu y} \, dy \, dx$

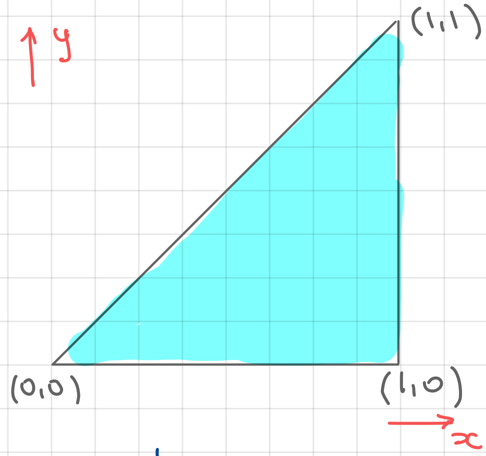
$$= \int_0^\infty \lambda e^{-\lambda x} \underbrace{\int_x^\infty \mu e^{-\mu y} dy}_{= P(Y>x)} dx = \int_0^\infty \lambda e^{-\lambda x} e^{-\mu x} dx = \int_0^\infty \lambda e^{-(\lambda+\mu)x} dx = \frac{\lambda}{\lambda+\mu} \int_0^\infty (\lambda+\mu) e^{-(\lambda+\mu)x} dx$$

$$= \frac{\lambda}{\lambda+\mu}$$

Joint distribution of $X_1, \dots, X_n$. Summary

Discrete	Continuous
Joint distribution	
Joint PMF	**Joint PDF**
$p(k_1, k_2, \dots, k_n) = P(X_1 = k_1, X_2 = k_2, \dots, X_n = k_n)$	$P((X_1, X_2, \dots, X_n) \in B) = \int \cdots \int_B f(x_1, \dots, x_n)\, dx_1 \cdots dx_n$
Marginal distribution	
Marginal PMF	**Marginal PDF**
$p_{X_j}(\ell) = \sum_{k_1} \cdots \sum_{k_{j-1}} \sum_{k_{j+1}} \cdots \sum_{k_n} p(k_1, \dots, k_{j-1}, \ell, k_{j+1}, \dots, k_n)$	$f_{X_j}(y) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_n)\, dx_1 \cdots dx_{j-1}\, dx_{j+1} \cdots dx_n$
Independence	
$p(k_1, \dots, k_n) = p_{X_1}(k_1) \cdots p_{X_n}(k_n)$	$f(x_1, \dots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$

Joint distribution and independence



Exercise Let X and Y be jointly continuous random variables with joint PDF

$$f(x,y) = \begin{cases} c\,xy, & 0 \le y \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

1) Find constant c

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy = c\int_0^1\int_0^x xy\,dy\,dx = 1$$

$$c \cdot \frac{1}{8} \Rightarrow c = 8$$

2) Find marginal densities of X and Y

3) Are X and Y independent? not independent

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^x 8 \cdot x \cdot y\,dy = 8 \cdot x \cdot \frac{x^2}{2} = 4x^3 \quad \text{for } 0 \le x \le 1$$

$$f_Y(y) = 4y(1-y^2), \quad 0 \le y \le 1$$

$f_X(x)\,f_Y(y) = f(x,y)$ for $0<x<y<$

Joint distribution and independence

Exercise: Let the joint distribution of random variables X and Y be given by the joint PMF $p(k,\ell) = P(X=k, Y=\ell)$

$k \backslash \ell$	0	1	2
-1	0.1	0.1	0.2
1	0.2	0.1	α

$$\sum_{k,\ell} p(k,\ell) = 1$$

1) Find unknown α

2) Compute the marginal PMFs of X and Y

k	-1	1
$P(X=k)$		

ℓ	0	1	2
$P(Y=\ell)$			

3) Are X and Y independent?

Joint distribution and independence

Example Let $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$

Suppose X and Y are independent.

Find the distribution of $Z = \min\{X, Y\}$

$$P(Z > t) = P(\min\{X,Y\} > t) = P(X > t, Y > t)$$

$$= P(X > t)\, P(Y > t)$$

$$= e^{-\lambda t}\, e^{-\mu t}$$

$$= e^{-(\lambda+\mu)t}$$

$$\Rightarrow Z \sim Exp(\lambda + \mu)$$