MATH 180A (Lecture A00)

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Today: Joint distribution. Independence

Next: ASV 8.1

Week 9:

- Homework 6 due Friday, March 10

Joint distribution (discrete case)

Def. (Joint PMF). Let $X_1, X_2, \dots, X_n$ be discrete random variable defined on the same probability space.

The joint probability mass function of $(X_1, \dots, X_n)$ is given by

Remark

Example Roll a fair die twice

$X_1 = \#$ of even numbers

$X_2 = \#$ of sixes

$p_{X_1, X_2}(k_1, k_2)$

$k_1 \backslash k_2$	0	1	2
0			
1			
2			

Expectation of a function of a random vector

Let $X_1, \dots, X_n$ be discrete random variables. Let $g: \mathbb{R}^n \to \mathbb{R}$

Then $E(g(X_1, \dots, X_n)) =$

where we sum over all possible values $k_1, \dots, k_n$ of $X_1, \dots, X_n$

Example (cont.) For each even number you get 1 dollar, and the sum is multiplied by the number of sixes.

$g(k_1, k_2) =$

$E(g(X_1, X_2)) =$

$=$

Marginal PMF

Let $p(k_1, \dots, k_n)$ be a joint PMF of random variables $X_1, \dots, X_n$. Then for any $1 \le j \le n$ the marginal PMF of X_j is given by

(fix j-th variable, sum over all other variables)

Example

$p(k_1, k_2)$

k_1 \ k_2	0	1	2
0	$\frac{9}{36}$	0	0
1	$\frac{12}{36}$	$\frac{6}{36}$	0
2	$\frac{4}{36}$	$\frac{4}{36}$	$\frac{1}{36}$

Q:

Joint distribution of continuous random variables

<u>Def.</u> Random variables $X_1, \dots, X_n$ are jointly continuous if there exists a function $f: \mathbb{R}^n \to \mathbb{R}$ such that

Function $f(x_1, \dots, x_n)$ is called the joint density

Joint density satisfies :

<u>Example</u> Consider random variables X and Y with joint density

Expectation of a function. Marginal PDF

Let $X_1, \dots, X_n$ be jointly continuous random variables with joint PDF f_X. Let $g: \mathbb{R}^n \to \mathbb{R}$ be a function of n variables.

Then

Def. Let f be the joint density of $X_1, \dots, X_n$. Then each random variable X_j has a (marginal) density

(fix j-th variable, integrate all other variables)

Example

Consider again random variables X, Y with joint density

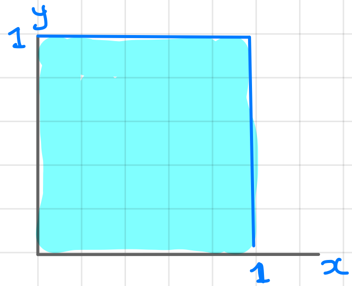
$$f(x,y) = \begin{cases} \frac{3}{2}(xy^2+y) & , \ 0 \le x \le 1, \ 0 \le y \le 1 \\ 0 & , \ \text{otherwise} \end{cases}$$

(i) $E(X^2Y) =$

$g(x,y) = x^2y$

(ii) $f_X(x) =$

$f_Y(y) =$



(iii) $P(X<Y) =$

Joint distribution and independence

Random variables $X_1, \dots, X_n$ defined on the same probability space are independent if for any $B_1, \dots, B_n \subset \mathbb{R}$

$$P(X_1 \in B_1, X_2 \in B_2, \cdots, X_n \in B_n) = P(X_1 \in B_1) P(X_2 \in B_2) \cdots P(X_n \in B_n)$$

Independence can be expressed in terms of PMF/PDF

Discrete case:

Let $p(k_1, \dots, k_n)$ be the joint PMF of discrete random variables $X_1, \dots, X_n$. Let $p_{X_j}(k) = P(X_j = k)$ be the marginal PMF of X_j. Then $X_1, \dots, X_n$ are independent if and only if

Examples

Example 1

Roll a fair die twice

X_1 = # even numbers

X_2 = # sixes

$K_1 \backslash K_2$	0	1	2	
0	$\frac{9}{36}$	0	0	$\frac{9}{36}$
1	$\frac{12}{36}$	$\frac{6}{36}$	0	$\frac{18}{36}$
2	$\frac{4}{36}$	$\frac{4}{36}$	$\frac{1}{36}$	$\frac{9}{36}$
	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$	

Are X_1 and X_2 independent?

Example 2

joint PMF of X_1, X_2

$K_1 \backslash K_2$	0	1	
0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
1	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$
	$\frac{1}{3}$	$\frac{2}{3}$	

Joint distributions and independence. Continuous case

<u>Thm</u>. Let $X_1, \dots, X_n$ be random variables defined on the same probability space. Assume that each X_j has PDF f_{X_j}.

(i) If the joint density of $X_1, \dots, X_n$ is equal to

$$f(x_1, \dots, x_n) =$$

then $X_1, \dots, X_n$ are independent

(ii) If $X_1, \dots, X_n$ are independent then

$$f(x_1, \dots, x_n) =$$

<u>Thm</u>. Let $X_1, \dots, X_n, X_{n+1}, \dots, X_{n+m}$ be independent random variables. Let $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^m \to \mathbb{R}$. Define $Y = f(X_1, \dots, X_n)$, $Z = g(X_{n+1}, \dots, X_{n+m})$. Then

Example

Let $X_1, \dots, X_n$ be independent random variables, $X_i \sim \text{Geom}(p_i)$. Define $Y = \min\{X_1, \dots, X_n\}$. Determin the distribution of Y.

$\{\min\{X_1, \dots, X_n\} = k\} =$

too complicated

Instead, $\{\min\{X_1, \dots, X_n\} > k\} =$

$P(Y > k) =$

$P(Y > k) =$

Joint distribution of $X_1, \dots, X_n$. Summary

Discrete	Continuous
Joint distribution	
Joint PMF	Joint PDF
$p(k_1, k_2, \dots, k_n)$ $= P(X_1 = k_1, X_2 = k_2, \dots, X_n = k_n)$	$P((X_1, X_2, \dots, X_n) \in B)$ $= \int \cdots \int_B f(x_1, \dots, x_n)\, dx_1 \cdots dx_n$
Marginal distribution	
Marginal PMF	Marginal PDF
$p_{X_j}(\ell) = \sum_{k_1} \cdots \sum_{k_{j-1}} \sum_{k_{j+1}} \cdots \sum_{k_n} p(k_1, \dots, k_{j-1}, \ell, k_{j+1}, \dots, k_n)$	$f_{X_j}(y) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_n)\, dx_1 \cdots dx_{j-1}\, dx_{j+1} \cdots dx_n$
Independence	
$p(k_1, \dots, k_n) = p_{X_1}(k_1) \cdots p_{X_n}(k_n)$	$f(x_1, \dots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$

Joint distribution and independence

Exercise Let the joint distribution of random variables X and Y be given by the joint PMF $p(k,\ell) = P(X=k, Y=\ell)$

$k \backslash \ell$	0	1	2
-1	0.1	0.1	0.2
1	0.2	0.1	α

1) Find unknown α

2) Compute the marginal PMFs of X and Y

k	-1	1
$P(X=k)$		

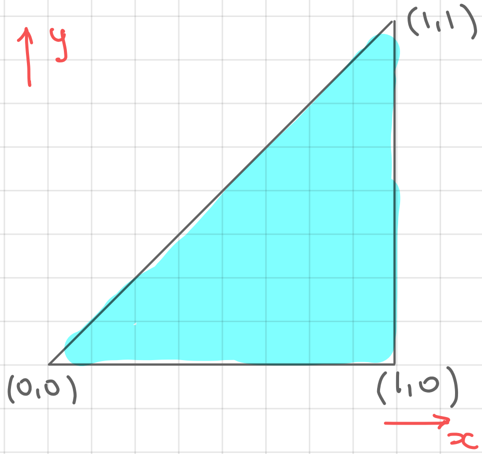
ℓ	0	1	2
$P(Y=\ell)$			

3) Are X and Y independent?

Joint distribution and independence

Exercise Let X and Y be jointly continuous random variables with joint PDF

$$f(x,y) = \begin{cases} c\,xy, & 0 \le y \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

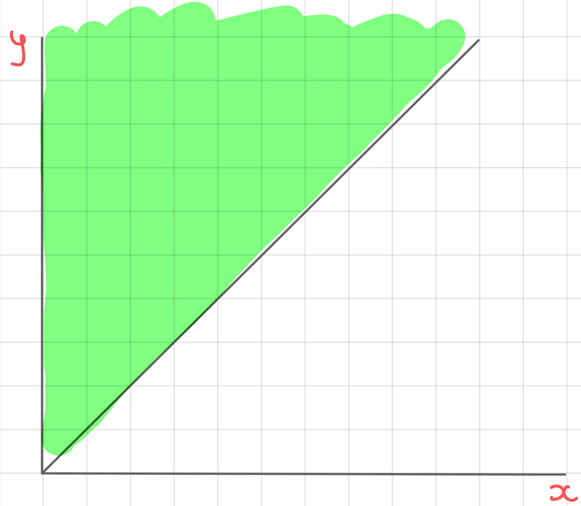


1) Find constant c
2) Find marginal densities of X and Y
3) Are X and Y independent?

$f_X(x) =$

$f_Y(y) =$

Joint distribution and independence



Example Let $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$

Suppose X and Y are independent.

1) Find the joint PDF of X and Y
2) Compute $P(X<Y)$

X takes values in $(0,+\infty)$, Y takes values in $(0,+\infty)$

(X,Y) take values in $\{(x,y) : x>0, y>0\}$

1) From the independence $f_{X,Y}(x,y) =$

2) $P(X<Y) = P((X,Y) \in \{(x,y) : x<y\}) =$

Joint distribution and independence

Example Let $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$

Suppose X and Y are independent.

Find the distribution of $Z = \min\{X, Y\}$

$$P(Z > t) = P(\min\{X, Y\} > t) =$$

Other things to remember

Expectation of a function of n random variables

Let $X_1, \dots, X_n$ be random variables, let $g: \mathbb{R}^n \to \mathbb{R}$

- if $X_1, \dots, X_n$ are discrete with joint PMF $p(k_1, \dots, k_n)$, then

$$E(g(X_1, \dots, X_n)) = \sum_{k_1 \dots k_n} g(k_1, \dots, k_n)\, p(k_1, \dots, k_n)$$

- if $X_1, \dots, X_n$ are continuous with joint PDF $f(x_1, \dots, x_n)$, then

$$E(g(X_1, \dots, X_n)) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g(x_1, \dots, x_n)\, f(x_1, \dots, x_n)\, dx_1 \cdots dx_n$$