MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

Today: Joint distribution

Next: ASV 8.1

Week 9:

- Homework 6 due Friday, March 10

Computing moments using MGF

Differentiate $M_X(t) = E(e^{tX})$ w.r.t. t

$$M_X'(t) = \frac{d}{dt} E(e^{tX}) = E\left(\frac{d}{dt} e^{tX}\right) = E\left(X e^{tX}\right)$$

$$M_X'(0) = E(X)$$

Differentiate again $\quad M_X''(0) = E(X^2)$

More generally

Thm. If $M_X(t)$ is bounded around $t=0$, then

$$E(X^n) = M_X^{(n)}(0)$$

No proof.

Alternatively, $\underbrace{E(e^{tX})}_{M_X(t)} = E\left(\sum_{n\geq 0}^{\infty} \frac{(tX)^n}{n!}\right) = \sum_{n=0}^{\infty} E(X^n) \frac{t^n}{n!} = \sum_{n\geq 0}^{\infty} M_X^{(n)}(0) \frac{t^n}{n!}$

Computing moments using MGF. Examples

- $P(X=1)=P(X=-1)=\frac{1}{2}$, $M_X(t)=\cosh(t)=\frac{e^{t}+e^{-t}}{2}$, $M_X'(t)=\sinh(t)$

$M_X''(t)=\cosh(t)\ \ldots$, $M_X^{(2k+1)}(t)=\sinh(t)$, $M_X^{(2k)}(t)=\cosh(t)$

$E(X^{2k})=M_X^{(2k)}(0)=1$, $E(X^{2k+1})=M_X^{(2k+1)}(0)=0$

- $X\sim N(0,1)$, $M_X(t)=e^{\frac{t^2}{2}}$

$$M_X(t)=e^{t^2/2}=\sum_{k=0}^{\infty}\frac{(t^2)^k}{2^k\cdot k!}=\sum_{k=0}^{\infty}\frac{t^{2k}}{2^k\cdot k!}=\sum_{k=0}^{\infty}\frac{(2k)!}{2^k\cdot k!}\frac{t^{2k}}{(2k)!}$$

$$=E(X^n)=\begin{cases}0, & \text{if } n \text{ is odd}\\ \frac{(2k)!}{2^k\cdot k!}, & \text{if } n=2k\end{cases} = (n-1)!!$$

$$\frac{(2k)!}{2^k\cdot k!}=\frac{1\cdot\cancel{2}\cdot3\cdot\cancel{4}\cdot5\cdot\cancel{6}\cdot\;---\;\cancel{(2k-2)}\cdot(2k-1)\cdot\cancel{2k}}{\cancel{2}\cdot\cancel{4}\cdot\cancel{6}\cdot\cancel{8}\;---\;\cdot\cancel{2\cdot(k-1)}\cdot\cancel{2k}}=1\cdot3\cdot5\cdot7\cdots(2k-1)$$

$$1\cdot3\cdot5\cdot7\cdots(2k-1) = (2k-1)!!$$

Distribution of a function of a random variable

Let X be a random variable, let g be a function defined on the range of X. We already know how to compute $E(g(X))$

$$E(g(X)) = \sum g(k)P(X=k) \qquad E(g(X)) = \int_{-\infty}^{+\infty} g(x) f_X(x)\,dx$$

Q: How to compute the PMF/PDF of $g(X)$?

Discrete case: (i) find the set of all possible values of $g(X)$

(ii) $$P(g(X)=\ell) = \sum_{k:\, g(k)=\ell} P(X=k)$$

Example

k	-1	0	1	2
$P(X=k)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

$g(x) = x^2$

ℓ	0	1	4
$P(X^2=\ell)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$$P(X^2=1) = P(X=1) + P(X=-1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Distribution of a function of a continuous random variable

Let X be a continuous random variable, $g: \mathbb{R} \to \mathbb{R}$

In order to compute the PDF of $g(X)$

(i) compute the CDF of $g(X)$

(ii) differentiate the CDF of $g(X)$

Example

Let $U \sim \mathrm{Unif}[0,1]$, let $g(x) = -\frac{1}{\lambda}\log(1-x)$ for $\lambda > 0$

Find the distribution of $Y = g(X)$.

(0) Range of (the possible values) of Y is $(0, +\infty)$

(i) $P(g(X) \le t) = P(-\frac{1}{\lambda}\log(1-X) \le t) = P(\log(1-X) \ge -\lambda t) = P(1-X \ge e^{-\lambda t})$

$= P(X \le 1 - e^{-\lambda t}) = 1 - e^{-\lambda t}$

(ii) $f_Y(t) = F_Y'(t) = \begin{cases} 0, & t < 0 \\ \lambda e^{-\lambda t}, & t > 0 \end{cases}$

Linear transformation of a continuous random variable

Example Let X be a continuous random variable with PDF $f_X(x)$ and CDF $F_X(x)$. Let $Y = aX + b$ with $b \in \mathbb{R}$, $a \neq 0$. Compute CDF and PDF of Y. $Y = g(X)$ with $g(x) = ax + b$

$$F_Y(y) = P(Y \leq y) = P(aX + b \leq y) = P(aX \leq y - b)$$

$$= \begin{cases} P\left(X \leq \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right) & \text{if } a > 0 \\ P\left(X \geq \frac{y-b}{a}\right) = 1 - F_X\left(\frac{y-b}{a}\right) & \text{if } a < 0 \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) = F_X'\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} & a > 0 \\ \frac{d}{dy}\left(1 - F_X\left(\frac{y-b}{a}\right)\right) = F_X'\left(\frac{y-b}{a}\right) \cdot \frac{1}{-a} & a < 0 \end{cases}$$

$$= F_X'\left(\frac{y-b}{a}\right) \cdot \frac{1}{|a|}$$

$$= f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{|a|}$$

General formula for a function of a continuous RV

Let X be a continuous random variable with PDF $f_X(x)$

If $g: \mathbb{R} \to \mathbb{R}$ is one-to-one, differentiable and $g'(x) = 0$ only on a finite set, then

$$f_{g(X)}(y) = f_X(g^{-1}(y)) \cdot \frac{1}{|g'(g^{-1}(y))|}$$

<u>Example</u> Let $X \sim N(0,1)$, $g: \mathbb{R} \to \mathbb{R}$, $g(x) = x^3$. Find PDF of X^3.

Let $Y = g(X) = X^3$. Y takes values on the whole $\mathbb{R}$.

If g is not one-to-one, split into intervals where g is one-to-one.

Random vectors

Until now we studied (mostly) individual random variables, one at a time, using various tools such as

PMF/PDF, CDF, expectation, variance, moments, MGF

We discussed some very simple models with finite/infinite number of random variables (independent trials)

New setting: random variables $X_1, X_2, X_3, \dots, X_n$, all defined on the same probability space (not necessarily independent)

$(\Omega, \mathcal{F}, P)$, $\underline{X} : \Omega \to \mathbb{R}^n$, $\underline{X}(\omega) = (X_1(\omega), X_2(\omega), \dots, X_n(\omega))$

$\underline{X} = (X_1, X_2, \dots, X_n)$

distribution of $\underline{X}$: $P(\underline{X} \in B)$ for all^{+} $B \subset \mathbb{R}^n$

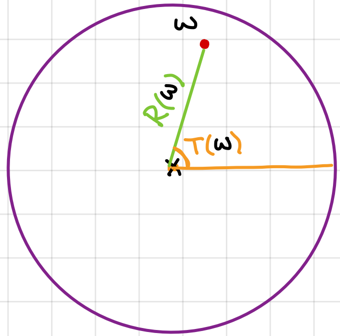
Example

- Choose a point ω from a unit disk

  $R(\omega)$ = distance to the center

  $T(\omega)$ = angle

  (R, T) is a random vector



- Roll a fair die 2 times

  X_1 = # of even numbers

  X_2 = # of sixes

  (X_1, X_2) is a random vector