MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

Today: Definition of probability

Next: ASV 1.2

Week 1:

- check the course website
- homework 1 (due Friday, January 20)
- join Piazza

Probability theory

The goal of probability theory is to build mathematical models of experiments with random outcomes

Random outcome = impossible to be predicted with certainty

1654: starting point, mathematical treatment of gambling problems (Fermat, Pascal)

1933: modern rigorous foundation of probability theory (Kolmogorov)

Warm-up problem

What is the probability that there are at least two student in this room having birthday on the same day (MM/DD)?

100 students, 365 possible birthday dates

Moral:

Axioms of probability

How to construct a mathematical model of an experiment with random outcome?

<u>Def.</u> Probability space is the triple $(\Omega, \mathcal{F}, P)$, where

- Ω is the

  ; we call it the sample space

- $\mathcal{F}$ is a

- P is a function that assigns to each event a real number and satisfies the following properties:

  (i)

  (ii)

  (iii)

Examples

We call function P that satisfies properties (i)-(iii)

a , or simply probability.

Example 1: Tossing a coin.

$\Omega =$, $\mathcal{F} =$, $P(\emptyset) =$, $P(\Omega) =$

$P(\{H\}) =$, $P(\{T\}) =$

$\{H\}$ and $\{T\}$ are disjoint, therefore, by (iii)

$P(\{H\}) + P(\{T\}) = P(\{H\} \cup \{T\}) =$

For any $\alpha \in [0,1]$ we have a different probability measure on Ω and $\mathcal{F}$.

Fair coin: , $P(\{H\}) = P(\{T\}) =$

Examples

Example 2: rolling a fair die

$\Omega =$, $\mathcal{F} =$

$P(\{1\}) = P(\{2\}) = \cdots = P(\{6\}) =$

What about the events? Take $A = \{2,4,6\} \subset \Omega$.

$P(A) =$

$B = \{2,3,5\}$: $P(B) =$

$C = \{3,6\}$: $P(C) =$

$A =$

$B =$

$C =$

$P(A \cup B) =$

$P(B \cap C) =$

Repeated experiments

What is the sample space if we toss the coin twice?

The outcome is a pair with

The collection of such pairs is called the Cartesian product of $\{H,T\}$ and $\{H,T\}$, denoted $\{H,T\} \times \{H,T\}$

$$\{H,T\} \times \{H,T\} = \{(H,H),(H,T),(T,H),(T,T)\}$$

sample space

More generally, for any sets $\Omega_1, \Omega_2, \dots, \Omega_k$

sample space if we perform experiment 1 with s.s. Ω_1

experiment 2 with s.s. Ω_2

$\vdots$

experiment k with s.s. Ω_k

Finite sample space

Consider a special case when $\#\Omega < \infty$. Then

$\Omega =$ for $n = \#\Omega$

Any event $A \subset \Omega$ is a finite union of $\{\omega_i\}$.
The singleton sets $\{\omega_1\}, \ldots, \{\omega_n\}$ are disjoint.

Therefore, if $A = \{a_1, \ldots, a_k\}$ for some $a_i \in \Omega$, then

$P(A) =$

What if additionally we have that $P(\{\omega_1\}) = \cdots = P(\{\omega_n\})$?

Uniform probability measure and random sampling

If Ω is finite, the uniform probability measure is defined by the following property:

$$\text{for each } \omega \in \Omega, \quad P(\{\omega\}) =$$

From (*) this implies that

$$\text{for any event } A, \quad P(A) =$$

This means that for such models calculating probabilities is reduced to counting.

<u>Example</u> Roll a fair dice twice. What is the probability that the sum is 4? $\Omega = \{(i,j) : 1 \le i,j \le 6\}$,

$A = \{(1,3),(2,2),(3,1)\}$, $\qquad\qquad P(A) =$

Uniform probability measure and random sampling

Example A fair coin is tossed 3 times.

$A = \{\text{at least two tails}\}$

$B = \{\text{exactly two tails}\}$

$\Omega = \{(x_1, x_2, x_3) : x_i \in \{H, T\}\}$, $\#\Omega =$

$A =$, $\# A =$

$B =$, $\# B =$

$P(A) =$, $P(B) =$