MATH 180A (Lecture A00)

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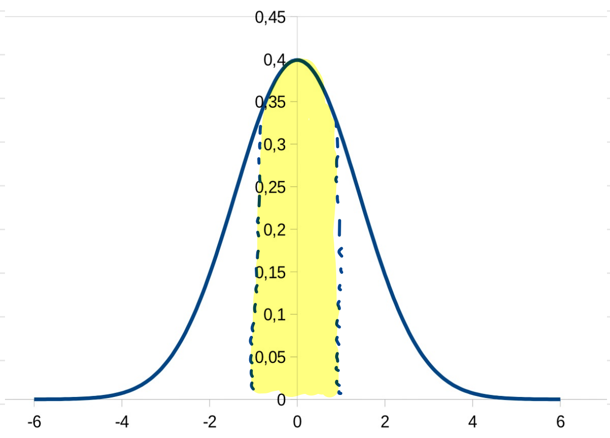
Today: Gaussian (Normal) distribution
Normal approximation
Next: ASV 4.1

Week 6:

- Homework 4 due Friday, February 17

CDF of $\mathcal{N}(0,1)$

Suppose $X \sim \mathcal{N}(0,1)$. What is $P(|X| \le 1)$?



$P(-1 \le X \le 1)$

$$= \int_{-1}^{1} \varphi(t)\,dt = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-t^2/2}\,dt$$

Cannot use the polar coordinate trick.

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt$$ - CDF of $X \sim \mathcal{N}(0,1)$

- no simple explicit formula
- table of values of $\Phi(x)$ (for $x \ge 0$)

Normal table of values (Appendix E in textbook)

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389

This table gives $P(Z \leq z)$ where $Z \sim N(0,1)$, $z = x_i + y_j$

Example $\Phi(0.91) = P(Z \leq 0.91) = P(Z \leq 0.9 + 0.01) \approx 0.8186$

Fact:

$P(Z > 0.24) =$

$P(-0.28 < Z < 0.59) =$

Normal table of values

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
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0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
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0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389

<u>Exercise</u> Let $Z \sim \mathcal{N}(0,1)$

Find $x_0 \in \mathbb{R}$ such that $P(|Z| > x_0) \approx 0.704$

$P(|Z| > x_0) =$

Mean and variance of $X \sim N(0,1)$

$$E(X) = \int_{-\infty}^{+\infty} t \, f_X(t) \, dt =$$

$$Var(X) = E(X^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} t^2 e^{-\frac{t^2}{2}} dt$$

General normal distribution $\quad N(\mu, \sigma^2)$

<u>Def</u> Let $\mu \in \mathbb{R}$ and $\sigma > 0$. Random variable X has normal (Gaussian) distribution with mean μ and variance σ^2 if the PDF of X is given by

$$f_X(x) =$$

We write

Using the density we can compute

$$E(X) = \qquad , \quad Var(X) =$$

"Gaussian distribution" = family of distributions

Relation between $X \sim N(\mu, \sigma)$ and $Z \sim N(0,1)$

Proposition Let $X \sim N(\mu, \sigma^2)$, $a \neq 0$, $b \in \mathbb{R}$.

Then

Using this proposition any Gaussian random variable can be written as a shifted and rescaled standart normal.

E.g., if $\sigma > 0$, $\mu \in \mathbb{R}$ and $Z \sim N(0,1)$, then

If $X \sim N(\mu, \sigma^2)$, then $E(X) =$; $Var(X) =$

If $X \sim N(\mu, \sigma^2)$, then

Example

Let $X \sim N(-3, 4)$

Find $P(X < 0.91)$; $P(X > 0.82)$; $P(-0.24 < X < 0.88)$

If $X \sim N(-3, 4)$, then , so

$P(X < 0.91) =$

$P(-0.24 < X < 0.88) =$

The message:

If we have independent and identically distributed random variables $X_1, X_2, \dots, X_n$ with $E(X_1) = \mu$, $Var(X_1) = \sigma^2$, then for any $a < b$

Today: $X_1 \sim Ber(p)$; Last lecture: general case

CLT for Bernoulli distribution (approximation of Bin)

If $X_i \sim Ber(p)$ are independent, then $X_1 + \cdots + X_n \sim Bin(n,p)$

$E(X_1) =$ $\qquad$ $Var(X_1) =$

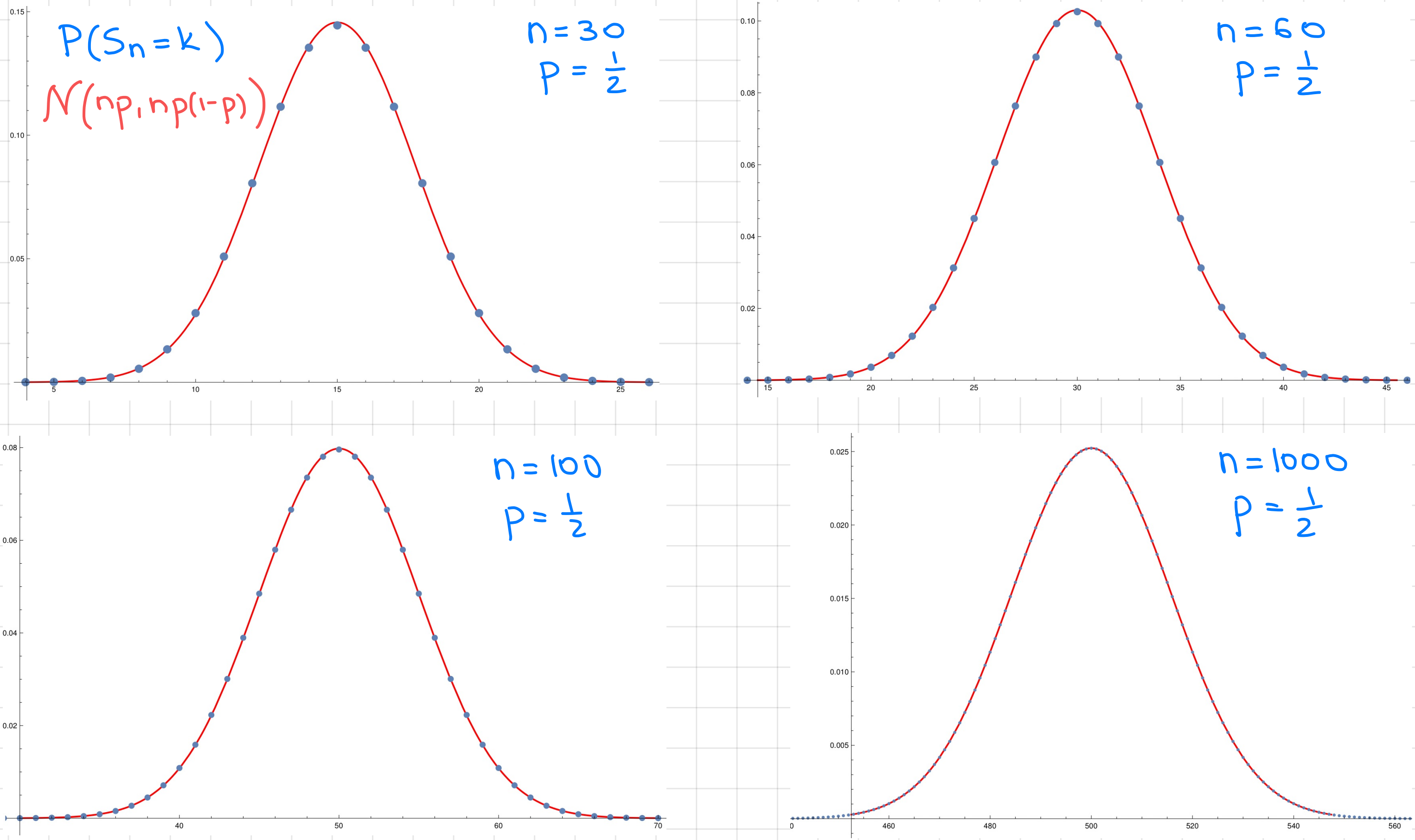
CLT for Bernoulli distribution:

Let $S_n \sim Bin(n,p)$, let $a<b$. Then

We can rewrite (*) using $\overline{S_n} := \frac{S_n}{n}$

CLT, approximation of Binomial distribution

Some numerics



Normal approximation. 3-sigma rule

We use the approximation of Bin(n,p) by the normal distribution if

In this case we can take

$$P\left(a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b\right) \approx \Phi(b) - \Phi(a)$$

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
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0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9958	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

In particular, in this case

- $P(|S_n - np| < \qquad) \approx \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.68$
- $P(|S_n - np| < \qquad) \approx \Phi(2) - \Phi(-2) = 2\Phi(2) - 1 = 0.95$
- $P(|S_n - np| < \qquad) \approx \Phi(3) - \Phi(-3) = 2\Phi(3) - 1 = 0.99$

CLT. Examples

Flipping a fair coin 10000 times

X = number of tails

Find (approximately) $P(4950 \le X \le 5050)$

z	0.00	0.01	0.02
0.0	0.5000	0.5040	0.5080
0.1	0.5398	0.5438	0.5478
0.2	0.5793	0.5832	0.5871
0.3	0.6179	0.6217	0.6255
0.4	0.6554	0.6591	0.6628
0.5	0.6915	0.6950	0.6985
0.6	0.7257	0.7291	0.7324
0.7	0.7580	0.7611	0.7642
0.8	0.7881	0.7910	0.7939
0.9	0.8159	0.8186	0.8212
1.0	0.8413	0.8438	0.8461
1.1	0.8643	0.8665	0.8686

$X \sim Bin(10\,000, \frac{1}{2})$,

$E(X) =$

$\sigma(X) =$

$P(4950 \le X \le 5050) =$

CLT. Examples

You win \$9 with probability $\frac{1}{20}$, lose \$1 with prob. $\frac{19}{20}$

Approximate the probability that you lost < 100 \$ after 400 games.

Denote by X the number of wins after 400 games

$X \sim \text{Bin}(400, \frac{1}{20})$. $\quad n \cdot p \cdot (1-p) =$

Total winnings after 400 games:

We have to compute

$$P(9X - (400 - X) > -100) =$$

Law of Large Numbers

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed, and let $E(X_1) = \mu \in \mathbb{R}$. Then

for any $\varepsilon > 0$

In particular, for $X_1 \sim Ber(p)$