MATH 180A (Lecture A00)

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Today: Expectation

Next: ASV 3.4

Week 5:

- Homework 3 due Friday, February 10
- Regrades of Midterm 1, HW 1, HW2 active on Gradescope until February 12, 11 PM

Expectation

<u>Def</u>. Let X be a discrete random variable with possible values $t_1, t_2, t_3, \ldots$. The expectation or expected value or mean of X is

$$E(X) := \sum_j t_j \cdot P(X = t_j)$$

weighted average

<u>Example</u> Let Y be $\mathrm{Ber}(p)$.

$$E(Y) = 1 \cdot p + 0 \cdot (1-p) = p$$

<u>Example</u> You toss a biased coin repeatedly until the first heads. How long do you expect it to take?

N = the time the first heads comes up, $N \sim \mathrm{Geom}(p)$

$$E(N) = \frac{1}{p}$$

Examples. Binomial

$S_n \sim Bin(n,p)$ ($S_n = X_1 + X_2 + \cdots + X_n$ for X_j independent $Ber(p)$)

$E(S_n) =$

Examples. Poisson

$X \sim \text{Poisson}(\lambda)$

$E(X) =$

Example A factory has, on average, 3 accidents per month. Estimate the probability that there will be exactly 2 accidents this month.

Examples

Toss a fair coin until tails comes up. If this is on the first toss, you win 2 dollars and stop. If heads comes up, the pot doubles and you continues. That is, if the first tails is on the k-th toss, you win 2^k dollars. What is your expected winnings?

Expectation of continuous random variables

X discrete, $X \in \{t_1, t_2, \dots\}$

$$E(X) = \sum_k t_k \, P(X = t_k)$$

X continuous ($P(X=t)=0$ for each $t \in \mathbb{R}$)

with density $f_X(t)$

<u>Example</u> Let $U \sim \text{Unif}([a,b])$, $f_U(t) = \begin{cases} \frac{1}{b-a}, & a \le t \le b \\ 0, & \text{otherwise} \end{cases}$

Example

Q. Shoot an arrow at a circular target of radius 1. What is the expected distance of the arrow from the center?

a) 1

b) $\frac{2}{3}$

c) $\frac{1}{2}$

d) $1/4$

e) 0

Expectation of continuous random variables

Example Consider function $f(t) = \begin{cases} 0, & t \le 1 \\ \frac{1}{t^2}, & t > 1 \end{cases}$

Is $f(t)$ a probability density?

$f(t) \ge 0$ $\qquad \int_{\mathbb{R}} f(t)\,dt =$

Suppose that X is a random variable with PDF $f_X = f$.

What is $E(X)$?

$$E(X) = \int_{-\infty}^{+\infty} t\, f_X(t)\,dt =$$

Expectations of functions of random variables

$$\Omega \xrightarrow[\text{random variable}]{X} \mathbb{R} \xrightarrow[\text{function}]{} \mathbb{R}$$

composition $g(X) = g \circ X$

<u>Example</u> $X \sim Bin(n,p)$ is the number of successes in n trials

$g(X) = \frac{X}{n}$ is the proportion of successful trials

$g(X) \in \{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n}{n} = 1\}$,

$E(g(X)) =$

<u>Proposition</u> For a discrete random variable X

$E(g(X)) =$

Note, by definition $E(g(X)) = \sum_t t\, P(g(X) = t)$

Expectations of functions of random variables

<u>Proposition</u> For a continuous random variable X with density f_X

$$E(g(X)) =$$

<u>Example</u> Let U be a uniform random variable on $[a,b]$

Then

$$E[U^2] =$$

$$f_U(t) = \begin{cases} \frac{1}{b-a}, & t \in [a,b] \\ 0, & \text{otherwise} \end{cases}$$

$$=$$

Important class of functions: $g(x) = x^n$

discrete: $E(X^n) = \sum_t t^n P(X=t)$, continuous: $E(X^n) = \int_{-\infty}^{\infty} t^n f_X(t)\,dt$

Expectations of functions of random variables

Example (Car accident / insurance example)

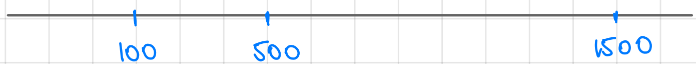
An accident causes Y dollars of damage to your car, where insurance deductible is 500 dollars.

$Y \sim \text{Unif}([100, 1500])$. What is the expected amount you pay?

X = amount you pay =
(neither discrete nor continuous)

$$E(X) = E(g(Y)) =$$

$$\min\{t, 500\} = \begin{cases} t, & t \in [100, 500] \\ 500, & t \in [500, 1500] \end{cases}$$



Variance

Definition The variance of a random variable X is

- first compute $\mu = E(X)$, then compute $E(g(X))$ with $g(t) = (t-\mu)^2$
- if X is discrete, $Var(X) =$
- if X is continuous, $Var(X) =$
-
- The square root of the variance is called

Variance. Example

Example $X \sim Ber(p)$, $E(X) = p$

$Var(X) =$

$=$

$=$

Example $U \sim Unif[a,b]$, $E(U) = \frac{a+b}{2}$

$Var(U) =$

Alternative formula for variance

Proposition. Let X be a random variable. Then

$$\mathrm{Var}(X) =$$

Proof (For continuous random variables) Let $\mu := E(X)$.

Then

$$\mathrm{Var}(X) = E((X-\mu)^2) = \int_{\mathbb{R}} (t-\mu)^2 f_X(t)\,dt$$

$$=$$

$$=$$

$$=$$

Example $X \sim \mathrm{Ber}(p)$, $E(X) = p$,

Variance

Variance is a measure of how "spread out from the mean" the distribution is.

Proposition Let X be a random variable with finite expectation $E(X)=\mu$. Then

Proof $(\Leftarrow)$ Exercise

$(\Rightarrow)$ (Assume X is discrete).

$0 = \mathrm{Var}(X) =$ $\Rightarrow$ For all t,

For all t, either or , so if

therefore,

Expectation and variance of $aX+b$

Let X be a random variable, and let $a, b \in \mathbb{R}$. Then

(i) $E(aX+b) =$

(ii) $Var(aX+b) =$ if $E(X)$ and $Var(X)$ exist

Proof (i) ← homework

(ii) $Var(aX+b) =$

$=$

$=$

$=$

$=$

Variance of geometric distribution

Let $X \sim Geom(p)$. We know that $E(X) = \frac{1}{p}$

$$E(X^2) = \sum_{k=1}^{\infty} k^2 P(X=k) = \sum_{k=1}^{\infty} k^2 \cdot p(1-p)^{k-1} =$$

$$=$$

$l = k-1 \quad =$

$$=$$

$$=$$

Random variables. Summary

Discrete	Continuous
Finite/countable set of possible values, $\sum_t P(X=t)=1$	Uncountable set of possible values, $\forall t\in\mathbb{R}$ $P(X=t)=0$
PMF: $p_X(t)=P(X=t)$ $P(X\in B)=\sum_{t\in B} p_X(t)$	PDF: $f_X:\mathbb{R}\to\mathbb{R}$ $P(X\in B)=\int_B f_X(t)\,dt$
CDF F_X is a step function	CDF F_X is a continuous function
Expectation: $E(X)=\sum_t t\,P(X=t)$	Expectation: $E(X)=\int_{\mathbb{R}} t\,f_X(t)\,dt$
$E(g(X))=\sum_t g(t)\,P(X=t)$	$E(g(X))=\int_{\mathbb{R}} g(t)\,f_X(t)\,dt$
Relation between CDF and PMF: magnitude of jump of F_X at t is $P(X=t)$	Relation between CDF and PDF: $f_X(t)=F_X'(t)$ on the intervals where F_X is differentiable

Random variables. Summary

F_X is (i) nondecreasing, (ii) right-continuous

(iii) $\lim_{t \to -\infty} F_X(t) = 0$, $\lim_{t \to +\infty} F_X(t) = 1$

Variance: $Var(X) = E\left((X - E(X))^2 \right) = E(X^2) - \left(E(X)\right)^2$

$E(aX + b) = aE(X) + b$, $Var(aX + b) = a^2 Var(X)$