MATH 180A (Lecture A00)

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Today: Expectation

Next: ASV 3.4

Week 5:

- Homework 3 due Friday, February 10
- Regrades of Midterm 1, HW 1, HW2 active on Gradescope until February 12, 11 PM

Rare events. Poisson distribution

$\forall x \in \mathbb{R} \quad \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$

Let $\lambda > 0$ and let X be a r.v. taking values in $\{0,1,2,\dots\}$.

X has Poisson distribution with parameter λ if

$$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{for } k \in \{0,1,2,\dots\}$$

We write $X \sim \text{Pois}(\lambda)$

Poisson distribution describes the probability that a "rare" event occurs k times after repeating the experiment (independent trials) "many" times.

Is this a probability distribution?

$$P(X=k) \geq 0, \quad \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1$$

λ gives the "expected number" of occurrances

Rare events. Poisson distribution

Let X be the number of successes in n independent trials with success probability $\frac{\lambda}{n}$, $\lambda > 0$.

Then $P(X = k) =$

What happens if $\qquad\qquad$ ($k \in \{0, 1, 2, \dots\}$ is fixed)?

Poisson distribution. Example

Observation: between 1875 and 1894 (20 years) in 14 units of Prussian army there were 196 deaths from horse kicks, distributed in the following way

# deaths per unit per year, k	# unit-years with k deaths	empirical probability	$P(X=k)$
0	144	0 51	
1	91	0.33	
2	32	0.11	
3	11	0.04	
4	2	0.01	
5+	0	0	
	total 280		

$P(X=0) = e^{-0.7}$

$P(X=1) = 0.7\, e^{-0.7}$

$P(X=2) = \frac{(0.7)^2}{2} e^{-0.7}$

Let is "expected number" of death per unit

Poisson distribution. Example

A 100 year storm is a storm magnitude expected to occur in any given year with probability $\frac{1}{100}$.
Over the course of a century, how likely is it to see at least 4 100 year storms?

Summary

Independent trials: the most important (discrete) probability distributions are:

- $\text{Ber}(p)$: $P(X=1)=p$, $P(X=0)=1-p$, $0 \le p \le 1$

(single trial with success probability p)

- $\text{Bin}(n,p)$: $P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$, $0 \le k \le n$

(number of successes in n independent trials with rate p)

- $\text{Geom}(p)$: $P(N=k) = (1-p)^{k-1} p$, $k = 1, 2, 3, \ldots$

(first successful trial in repeated independent trials with rate p)

- $\text{Poisson}(\lambda)$: $P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}$, $k = 0, 1, 2, \ldots$ $\lambda > 0$

(approximates $\text{Bin}(n, \frac{\lambda}{n})$, number of rare events in many trials)

Expectation

Example Toss a fair coin 1000 times, and record the sequence of outcomes

What if the coin is biased $P(X_j = 1) = p$, $P(X_j = 0) = 1-p$?

Def. Let X be a discrete random variable with possible values $t_1, t_2, t_3 \dots$

Expectation

Q: Is the expectation $E(X)$ the value that X is equal to most often?

(a) Yes, always

(b) No, not generally

Example Let X be the number rolled on a fair die.

Example Let Y be $Ber(p)$.

Expectation

Example You toss a biased coin repeatedly until the first heads. How long do you expect it to take?

Examples. Binomial

$S_n \sim \mathrm{Bin}(n,p)$ ($S_n = X_1 + X_2 + \cdots + X_n$ for X_j independent $\mathrm{Ber}(p)$)

$E(S_n) =$

Examples. Poisson

$X \sim \text{Poisson}(\lambda)$

$E(X) =$

<u>Example</u> A factory has, on average, 3 accidents per month. Estimate the probability that there will be exactly 2 accidents this month.

Examples

Toss a fair coin until tails comes up. If this is on the first toss, you win 2 dollars and stop. If heads comes up, the pot doubles and you continues. That is, if the first tails is on the k-th toss, you win 2^k dollars. What is your expected winnings?