MATH 180A (Lecture A00)

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Today: Independent trials

Next: ASV 3.3

Week 5:

- Homework 3 due Friday, February 10

Independent random variables

A collection $X_1, X_2, \ldots, X_n$ of random variables defined on the same sample space are independent if for any $B_1, B_2, \ldots, B_n \subset \mathbb{R}$, the events

i.e.,

Special case: if X_j are discrete random variables, it suffices to check the simpler condition

for any real numbers $t_1, t_2, \ldots, t_n$

Example Let $X_1, X_2, \ldots, X_n$ be fair coin tosses, $H \sim 1$, $T \sim 0$

$P(X_1 = t_1, X_2 = t_2, \ldots, X_n = t_n) =$ $= P(X_1 = t_1) \cdots P(X_n = t_n)$

Bernoulli distribution

Experiments can have numerical observables, but sometimes you only observe whether there is

success or failure

We model this with a random variable X taking value 1 with probability p, and value 0 with probability $1-p$

In practice, we usually repeat the experiment many times, making sure to use the same setup each trial. The previous trials do not influence the future ones.

Independent trials. Binomial distribution

Let $X_1, X_2, \dots, X_n$ be independent Ber(p) random variables

E.g. $P(X_1 = 0, X_2 = 1, X_3 = 1, X_4 = 0, X_5 = 0, X_6 = 0)$

$=$

$=$

Run n independent trials, each with success probability p,

$$X_1, X_2, \cdots, X_n \sim \text{Ber}(p)$$

Let $S_n = \#$ successful trials

What is the distribution of S_n?

$P(S_n = k) = P(\{\text{exactly } k \text{ of the } n \text{ trials are successful}\})$

$0 \le k \le n$

$=$

If , # of Heads in n tosses

Independent trials. Binomial distribution

Example Roll a fair die 10 times. What is the probability that 6 comes up at least 3 times?

success

$X_1, X_2, \ldots, X_{10} \sim$

$P(S_{10} \geq 3) =$

$=$

$=$

What is the probability that no 6 is rolled in the 10 rolls?

$P(S_{10} = 0) =$

First success time. Geometric distribution

Keep rolling. Let N denote the first roll where a 6 appears. N is a random variable.

What is the distribution of N?

N = first success in repeated independent trials (success rate p).

Model trials with (unlimited number of) independent $\mathrm{Ber}(p)$'s

$$\{N=k\} = \{X_1=0, X_2=0, \ldots, X_{k-1}=0, X_k=1\}$$

$$P(N=k) =$$

$$=$$

Geometric Distribution Geom (p) on $\{1,2,3,\ldots\}$. Is it?

Rare events. Poisson distribution

Let $\lambda > 0$ and let X be a r.v. taking values in $\{0, 1, 2, \dots\}$.
X has Poisson distribution with parameter λ if

$$P(X = k) = \qquad\qquad \text{for } k \in \{0, 1, 2, \dots\}$$

We write

Poisson distribution describes the probability that a "rare" event occurs k times after repeating the experiment (independent trials) "many" times.

Is this a probability distribution?

$P(X=k) \geq 0$,

λ gives the "expected number" of occurrances

Poisson distribution. Example

Observation: between 1875 and 1894 (20 years) in 14 units of Prussian army there were 196 deaths from horse kicks, distributed in the following way

# deaths per unit per year, k	# unit-years with k deaths	empirical probability	P(X=k)
0	144		
1	91		
2	32		
3	11		
4	2		
5+	0		

total 280

Let   is "expected number" of death per unit

Poisson distribution. Example

A 100 year storm is a storm magnitude expected to occur in any given year with probability $\frac{1}{100}$.
Over the course of a century, how likely is it to see at least 4 100 year storms?

Summary

Independent trials: the most important (discrete) probability distributions are:

- $Ber(p)$: $P(X=1)=p, \quad P(X=0)=1-p, \quad 0 \le p \le 1$

(single trial with success probability p)

- $Bin(n,p)$: $P(S_n=k)=\binom{n}{k}p^k(1-p)^{n-k}, \quad 0 \le k \le n$

(number of successes in n independent trials with rate p)

- $Geom(p)$: $P(N=k)=(1-p)^{k-1}p, \quad k=1,2,3,\ldots$

(first successful trial in repeated independent trials with rate p)

- $Poisson(\lambda)$: $P(X=k)=e^{-\lambda}\frac{\lambda^k}{k!}, \quad k=0,1,2,\ldots \quad \lambda>0$

(approximates $Bin(n,\frac{\lambda}{n})$, number of rare events in many trials)