MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

Today: Independent trials

Next: ASV 2.4-2.5

Week 3:

- Homework 3 due Friday, February 9

Properties of the CDF

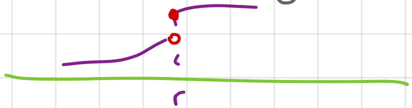
$F_X(r) = P(X \le r)$

(1) Monotone increasing: $s < t$, then $F_X(s) \le F_X(t)$

(2) $\lim_{r \to -\infty} F_X(r) = 0$, $\lim_{r \to +\infty} F_X(r) = 1$

(3) The function F_X is right-continuous:

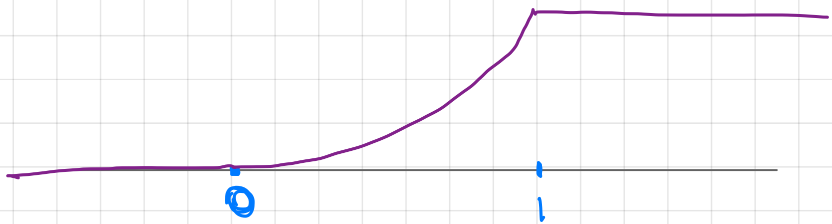
$$\lim_{t \to r_+} F_X(t) = F_X(r)$$



Corollary: If X is a continuous random variable, F_X is a continuous function

Example Shoot an arrow at a circular target of radius 1 (choose point from unit disk uniformly at random)

$$F_X(r) = \begin{cases} 0, & r \le 0 \\ r^2, & 0 \le r \le 1 \\ 1, & r \ge 1 \end{cases}$$



Cumulative distribution function (CDF)

Summary: For any random variable X, $F_X(r) = P(X \leq r)$

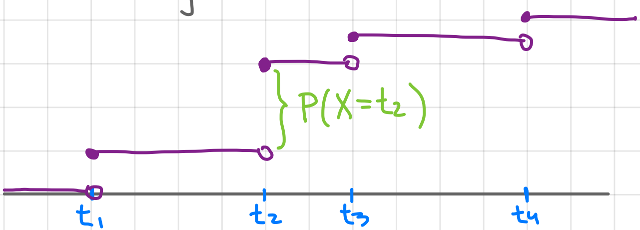
(1) Monotone increasing: $s \leq t \Rightarrow F_X(s) \leq F_X(t)$

(2) $\lim\limits_{r \to -\infty} F_X(r) = 0$, $\lim\limits_{r \to +\infty} F_X(r) = 1$

(3) Right-continuous: $\lim\limits_{t \to r_+} F_X(t) = F_X(r)$

Discrete random variable

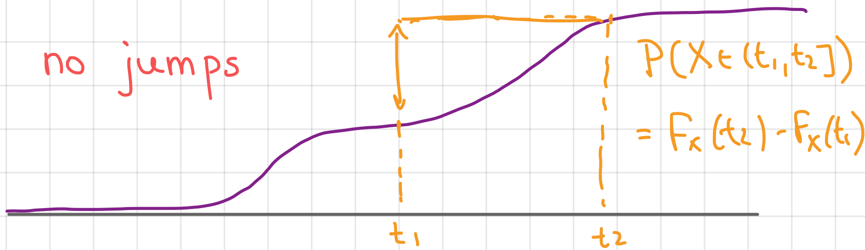
Finite or countable set of values with $t_1, t_2, \ldots$, $P(X = t_j) > 0$ and $\sum\limits_j P(X = t_j) = 1$



Continuous random variable

For each real number t, $P(X = t) = 0$

Because (1) and (3) this implies that F_X is continuous



Densities (PDF)

Some continuous random variables have probability densities. This is the infinitesimal version of the probability mass function.

X discrete, $X \in \{t_1, t_2, \dots\}$

$p_X(t) = P(X=t)$

probability mass function

$$P(X \in A) = \sum_{t_k \in A} P(X = t_k)$$

$$= \sum_{t_k \in A} p_X(t_k)$$

$p_X(t) \geq 0$, $\sum_t p_X(t) = 1$

X continuous

$P(X=t)=0$ for all $t \in \mathbb{R}$

probability density function $f_X(t)$

s.t. $P(X \in A) = \int_A f_X(t)\,dt$

e.g. $P(X \leq r) = \int_{-\infty}^{r} f_X(t)\,dt$

$f_X(t) \geq 0$, $\int_{-\infty}^{+\infty} f_X(t)\,dt = 1$

Densities (PDF)

Example Shoot an arrow at a circular target of radius 1.

X = distance from center

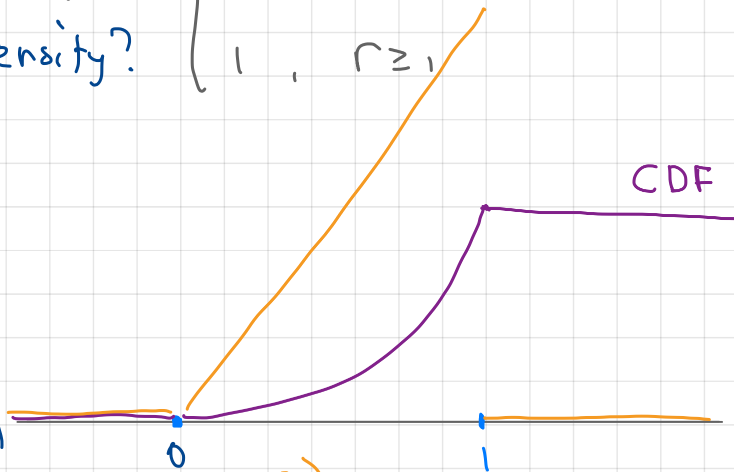
$$\int_{-\infty}^{r} f_X(t)\,dt \overset{?}{=} P(X \in (-\infty, r]) = F_X(r) = \begin{cases} 0, & r \le 0 \\ r^2, & 0 \le r \le 1 \\ 1, & r \ge 1 \end{cases}$$

can we find the density?

$$\frac{d}{dr}\left(\int_{-\infty}^{r} f_X(t)\,dt\right) = \frac{d}{dr} F_X(r)$$

FTC $\quad \| \quad\quad\quad\quad \|$

$$f_X(r) = \begin{cases} 0, & r \le 0 \\ 2r, & 0 < r < 1 \\ 0, & r \ge 1 \end{cases}$$



$$P(X \in [0.2, 0.5] \cup [0.9, 1.1]) = \underbrace{\int_{0.2}^{0.5} f_X(t)\,dt}_{= F_X(0.5) - F_X(0.2)} + \int_{0.9}^{1.1} f_X(t)\,dt$$

PDF : existence

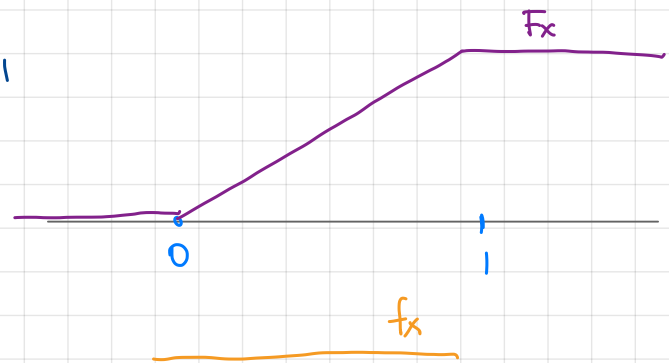
Thm : If F_X is continuous and (piecewise) differentiable, then X has density $f_X = F_X'$

Proof: Follows from FTC ∎

Example Let X = random number chosen uniformly on $[0,1]$

We have seen that in this case $P(X \in [s,t]) = t-s$, $0 \le s < t \le 1$

$$F_X(r) = P(X \le r) = \begin{cases} 0 & , r \le 0 \\ P(X \in [0,r]) = r & , 0 \le r \le 1 \\ 1 & , r > 1 \end{cases}$$



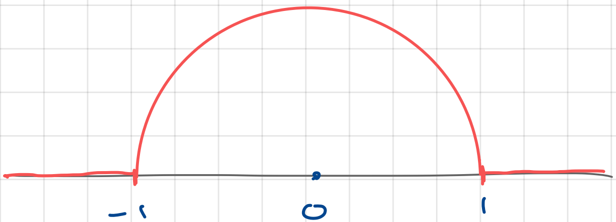
$$f_X(r) = \begin{cases} 0, & r \le 0 \\ 1, & 0 < r < 1 \\ 0, & r \ge 1 \end{cases}$$

$X \sim Unif([0,1])$

$$Z \sim Unif([a,b]) \to f_Z(t) = \begin{cases} \frac{1}{b-a} & , a \le t \le b \\ 0 & , \text{otherwise} \end{cases}$$

PDF

Example Let $f(t) = \begin{cases} c\sqrt{1-t^2}, & |t| \le 1 \\ 0, & \text{otherwise} \end{cases}, \quad c > 0$



Q: (When) Is $f(t)$ a PDF of some random variable?

- $f \ge 0$ ✓ ok
- $1 = \int_{-\infty}^{+\infty} f(t)\,dt = \int_{-1}^{1} c\sqrt{1-t^2}\,dt = c \cdot \frac{\pi}{2}, \quad c = \frac{2}{\pi}$

f is a PDF iff $c = \frac{2}{\pi}$, $f(t) = \frac{2}{\pi}\sqrt{1-t^2}\,\mathbb{1}_{|t| \le 1}$

Question

Your car is in a minor accident. The damage repair cost is a random number between 100 and 1500 dollars. Your insurance deductible is 500 dollars.

Z = your out of pocket expenses

Question: The random variable Z is

(a) continuous

(b) discrete

(c) neither

(d) both

$X \sim \mathcal{U}nif([100, 1500])$, $Z = \min(X, 500)$

$$f_X(t) = \begin{cases} \frac{1}{1500-100}, & 100 \leq t \leq 1500 \\ 0, & \text{otherwise} \end{cases}$$

$$P(Z=500) = P(X \geq 500) = \int_{500}^{1500} \frac{1}{1400}\,dt = \frac{5}{7} > 0$$

BUT if $r < 500$

$$P(Z=r) = P(X=r) = 0$$