MATH 10C: Calculus III (Lecture B00)

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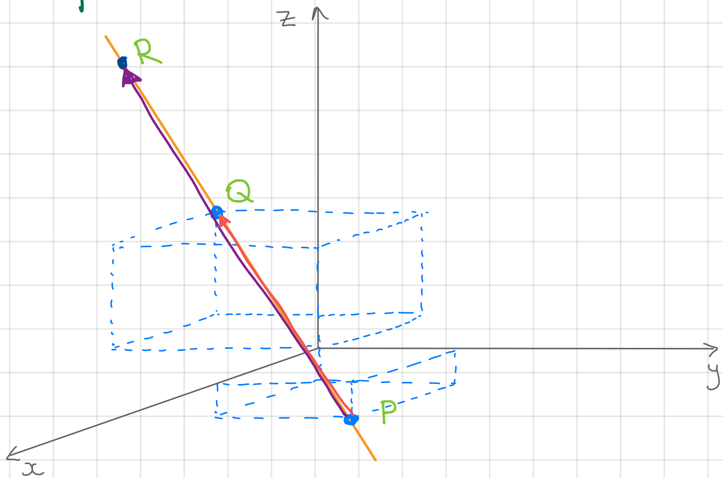
Today: Equations of lines and planes

Next: Strang 3.1

Week 3:

- homework 2 (due Monday, October 10)

Equation for a line in space



To describe a line in $\mathbb{R}^3$ we must know either

(a) two points on the line,

or (b) one point and direction.

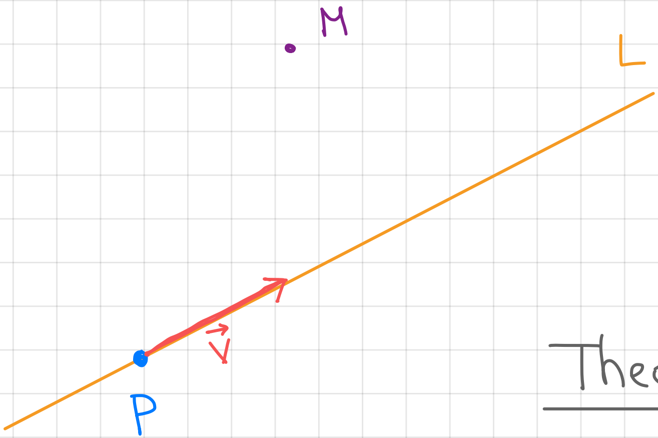
Thm 2.11 (Parametric and symmetric eqs. of a line)

A line parallel to vector $\vec{v} = \langle a, b, c \rangle$ and passing through $P = (x_0, y_0, z_0)$ can be described by the following parametric equations: $x = x_0 + ta, \; y = y_0 + tb, \; z = z_0 + tc \;, \; t \in \mathbb{R}$

If a, b and c are all nonzero, L can be described by the symmetric equation $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$

Distance between a point and a line

Consider the line L through point P with direction vector $\vec{v}$.
Suppose M is not on the line. What is the distance between L and M?



Theorem 2.12. Let L be a line passing through P with direction vector $\vec{v}$. If M is any point not on L, then

Distance between a point and a line

Example

Find the distance between $M = (3,2,1)$ and the line $\frac{x-5}{2} = \frac{y+2}{2} = -z$

Identify a point on the line.

Identify the direction vector of the line:

Compute

Finally,

Relationships between lines in $\mathbb{R}^3$

Let L_1 and L_2 be two lines in $\mathbb{R}^3$. Then the following four possibilities exist:

		L_1 and L_2 share a common point	
		YES	NO
Direction vectors of L_1 and L_2 are parallel	YES		
	NO		

Relationships between lines in $\mathbb{R}^3$

Example

L_1: direction vector $\vec{v}_1 = \langle 1, 2, 0 \rangle$, passing through $P_1 = (0,0,1)$

L_2: direction vector $\vec{v}_2 = \langle -3, -6, 0 \rangle$, passing through $P_2 = (1,2,3)$

L_3: direction vector $\vec{v}_3 = \langle 1, -1, 1 \rangle$, passing through $P_3 = (-1,4,-1)$

1. L_1 and L_2 $\vec{v}_1$ parallel to $\vec{v}_2$, , therefore,

   L_1 and L_2 are either

   Write equations for L_1:

Relationships between lines in $\mathbb{R}^3$

Example ② L_1 and L_3

(i) $\vec{v}_1 = \langle 1, 2, 0 \rangle$, $\vec{v}_3 = \langle 1, -1, 1 \rangle$. Are $\vec{v}_1$ and $\vec{v}_3$ parallel?

Parallel if and only if

$\left\{ \phantom{\begin{array}{c} a \\ b \\ c \end{array}} \right.$

this system has no solutions, so

direction vectors are

L_1 and L_3 are

(ii) Do L_1 and L_3 have a point in common?

If $Q = (x, y, z)$ belongs to both L_1 and L_3, then the coordinates of Q must satisfy both equations

Relationships between lines in $\mathbb{R}^3$

$$\begin{cases} x = t, \\ y = 2t, \\ z = 1, \end{cases} \quad \text{and} \quad \begin{cases} x = -1 + s \\ y = 4 - s \\ z = -1 + s \end{cases} \quad \text{for some } s, t \in \mathbb{R}$$

Equate the right-hand sides of the above equations

$$\left\{ \right.$$

If this system has a solution then L_1 and L_3 intersect

From the last equation we have . Substituting into the first two equations gives

Relationships between lines in $\mathbb{R}^3$

Example ③ : L_2 and L_3

L_2: direction vector $\vec{v}_2 = \langle -3, -6, 0 \rangle$, passing through $P_2 = (1, 2, 3)$

L_3: direction vector $\vec{v}_3 = \langle 1, -1, 1 \rangle$, passing through $P_3 = (-1, 4, -1)$

Since $\vec{v}_2$ and $\vec{v}_3$ are not parallel, L_2 and L_3 are either intersecting or skew. We have to check if L_2 and L_3 have a point in common.

L_2: $\Bigg\{$ $\qquad$ L_3: $\Bigg\{$ $\qquad$ Equate: $\Bigg\{$

$\rightarrow \Bigg\{ \qquad \Bigg\} \qquad \Bigg\{$

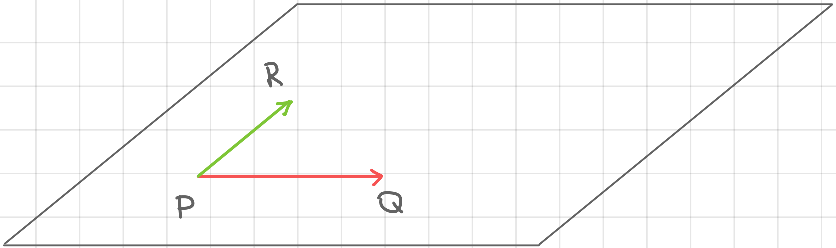
Planes

Two points determine a line: for any two points P, Q (in $\mathbb{R}^2$ or $\mathbb{R}^3$) there exists a unique line passing through P and Q. A point X is in the line through P and Q if $\vec{PX}$ is a multiple of $\vec{PQ}$, i.e., $\vec{PX} = t\,\vec{PQ}$ for some $t \in \mathbb{R}$.

Three points (that do not all lie on the same line) determine a plane: for any three points P, Q and R in $\mathbb{R}^3$ that do not all lie on the same line, there exists a unique plane that passes through these three points.

A point X is in the plane passing through P, Q and R if

Equation of a plane



Another way to describe a plane is by identifying

. If P is a point in the plane and vector $\vec{n}$ is orthogonal to the plane (called the normal vector) then point X is in this plane if and only if

Equation of a plane

Consider a plane containing point $P = (x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$. Then point $X = (x, y, z)$ belong to this plane if and only if

(*)

If we denote , then (*) becomes

Suppose that we know the coordinates of three points P, Q, R in the plane. How can we find a normal vector to this plane?

Example

Write the vector equation for the plane containing points $P=(1,1,0)$, $Q=(-2,1,1)$, $R=(0,0,1)$

Compute the normal vector to the plane

Point $X=(x,y,z)$ is in the plane if

, or equivalently