MATH 10C: Calculus III (Lecture B00)

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Today: Equations of lines and planes

Next: Strang 3.1

Week 2:

- homework ~~1~~ 2 (due Monday, October ~~3~~ 10)
- survey on Canvas Quizzes (due Friday, October 7)

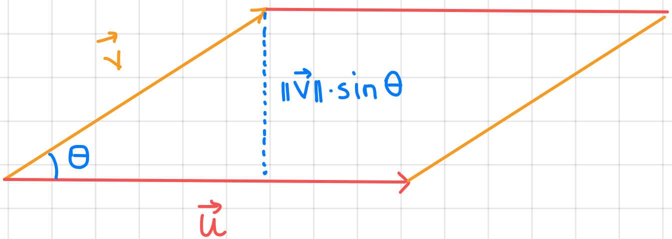
Cross product

Summary: Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^3$.

Then $\vec{u}\times\vec{v}$ is a vector in $\mathbb{R}^3$ such that

- $\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$ (right-hand rule)
- $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\sin\theta$ with θ = angle between $\vec{u}$ and $\vec{v}$

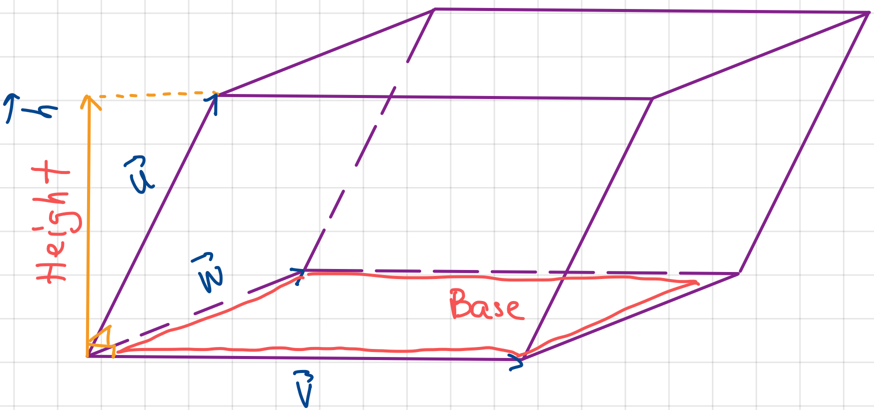
Consider a parallelogram spanned by vectors $\vec{u}$ and $\vec{v}$



$$\text{Area}(\square) = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\sin\theta$$
$$= \|\vec{u}\times\vec{v}\|$$

Conclusion: magnitude of $\vec{u}\times\vec{v}$ is equal to the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$

Volume of a parallelepiped



Three-dimensional prism with six facets that are each parallelograms.

Volume = (Area of the base) × Height $= |\vec{u} \cdot (\vec{v} \times \vec{w})|$

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$, consider a parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$.

Area of the base $= \|\vec{v} \times \vec{w}\|$

Height $= \|\text{proj}_{\vec{v}\times\vec{w}} \vec{u}\| = \dfrac{|\vec{u} \cdot (\vec{v} \times \vec{w})|}{\|\vec{v} \times \vec{w}\|}$

$$\|\vec{v} \times \vec{w}\| \cdot \frac{|\vec{u} \cdot (\vec{v} \times \vec{w})|}{\|\vec{v} \times \vec{w}\|}$$

$$= |\vec{u} \cdot (\vec{v} \times \vec{w})|$$

Volume of a parallelepiped

<u>Definition</u> The triple scalar product of $\vec{u}, \vec{v}$ and $\vec{w}$ is given by $\vec{u} \cdot (\vec{v} \times \vec{w})$

<u>Theorem 2.10</u> The volume of a parallelepiped given by vectors $\vec{u}, \vec{v}, \vec{w}$ is the absolute value of the triple scalar product $V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$

<u>Example</u> Find the volume of the parallelepiped with adjacent edges (spanned by) $\vec{u} = \langle -1, -2, 1 \rangle$, $\vec{v} = \langle 4, 3, 2 \rangle$, $\vec{w} = \langle 0, -5, -2 \rangle$

$\vec{v} \times \vec{w} = \langle 4, 8, -20 \rangle$, $\vec{u} \cdot (\vec{v} \times \vec{w}) = \langle -1, -2, 1 \rangle \cdot \langle 4, 8, -20 \rangle = -4 - 16 - 20 = -40$

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})| = |-40| = 40$$

Summary

Dot (scalar) product: $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

- characterizes the angle $0 \le \theta \le \pi$ between $\vec{u}$ and $\vec{v}$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$$

Cross (vector) product: $\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2)\vec{i} - (u_1 v_3 - u_3 v_1)\vec{j} + (u_1 v_2 - u_2 v_1)\vec{k}$

- gives a vector that is orthogonal to both $\vec{u}$ and $\vec{v}$
- its length give the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$ $\quad \|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \cdot \sin\theta$

Triple scalar product of $\vec{u}$, $\vec{v}$ and $\vec{w}$: $\vec{u} \cdot (\vec{v} \times \vec{w})$

- its absolute value gives the volume of the parallelepiped spanned by $\vec{u}$, $\vec{v}$ and $\vec{w}$.

Last remark

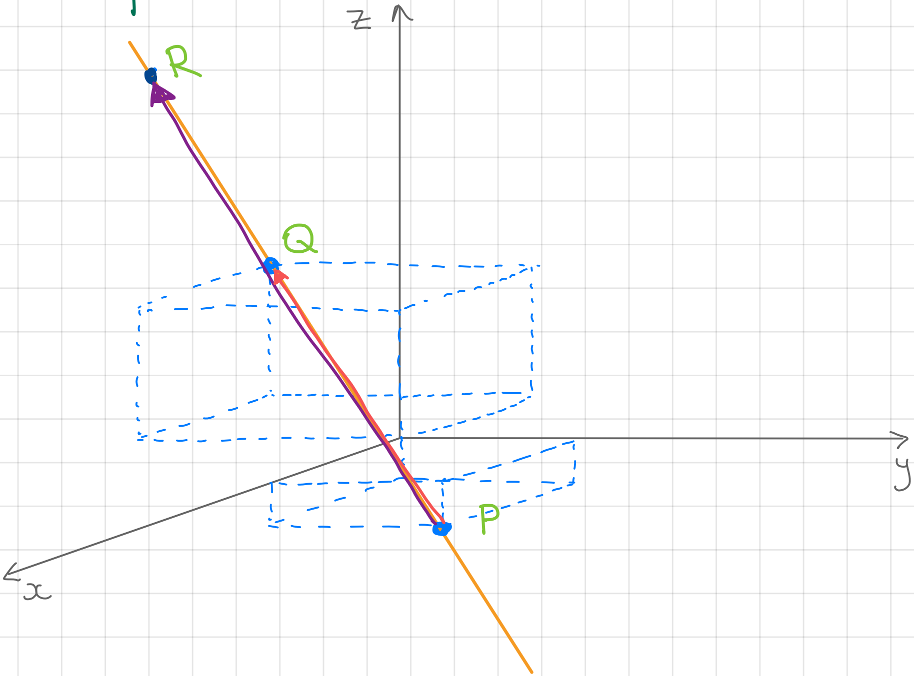
If you know how to compute the determinant of a 3×3 matrix, then the cross product of $\vec{u}=(u_1,u_2,u_3)$ and $\vec{v}=(v_1,v_2,v_3)$ can be computed as

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{i}\,(u_2v_3-u_3v_2)-\vec{j}\,(u_1v_3-u_3v_1)+\vec{k}\,(u_1v_2-u_2v_1)$$

Similarly, the triple scalar product of $\vec{u}=(u_1,u_2,u_3)$, $\vec{v}=(v_1,v_2,v_3)$ and $\vec{w}=(w_1,w_2,w_3)$ can be computed as

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1(v_2w_3-v_3w_2)-u_2(v_1w_3-v_3w_1)+u_3(v_1w_2-v_2w_1)$$

Equation for a line in space



To describe a line in $\mathbb{R}^3$ we must know either (a) two points on the line, or (b) one point and direction.

Let L be a line passing through points P and Q.

Point R belongs to L if $\vec{PR}$ is parallel to $\vec{PQ}$, i.e., either $\vec{PR}$ has the same direction as $\vec{PQ}$, or $\vec{PR}$ has direction opposite to $\vec{PQ}$ (or $\vec{PR} = \vec{0}$).

Equation for a line in space

Vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if

$$\vec{u} = k\vec{v} \quad \text{for some } k \in \mathbb{R}$$

(by convention $\vec{0}$ is parallel to all vectors)

Given two distinct points P and Q, the line through P and Q is the collection of points R such that

$$\overrightarrow{PR} = t\,\overrightarrow{PQ} \quad \text{for a real number } t \in \mathbb{R}$$

Similarly, given point P and vector $\vec{v}$, the line through P with direction vector $\vec{v}$ is the collection of points R such that

$$\overrightarrow{PR} = t\,\vec{v} \quad \text{for a real number } t \in \mathbb{R} \qquad (*)$$

Equation for a line in space

Let $P=(x_0, y_0, z_0)$, $R=(x,y,z)$ and $\vec{v}=\langle a,b,c\rangle$. Then

(*) implies

$$\vec{PR} = \langle x-x_0, y-y_0, z-z_0\rangle = \langle ta, tb, tc\rangle \qquad (**)$$

By equating components, we get that the coordinates of R (point on the line) satisfy the equations

parametric equations of a line
$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}, \quad t \in \mathbb{R} \qquad \frac{x-x_0}{a} = t, \quad \frac{y-y_0}{b} = t, \quad \frac{z-z_0}{c} = t \qquad (***)$$

If we denote $\vec{r} := \langle x,y,z\rangle$ and $\vec{r}_0 := \langle x_0, y_0, z_0\rangle$, then from (**)

$$\vec{r} = \vec{r}_0 + t\vec{v} \quad \text{(vector equation of a line)} \quad (****)$$

Equation for a line in space

If a, b and c are all nonzero, we can rewrite (***)

$$\frac{x-x_0}{a}=t, \quad \frac{y-y_0}{b}=t, \quad \frac{z-z_0}{c}=t,$$

which (since t can be any real number) is equivalent to

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$ symmetric equations of a line (*****)

Thm 2.11 (Parametric and symmetric eqs. of a line)

A line parallel to vector $\vec{v}=\langle a,b,c\rangle$ and passing through $P=(x_0,y_0,z_0)$ can be described by the following parametric equations: $x=x_0+ta,\ y=y_0+tb,\ z=z_0+tc\ ,\ t\in\mathbb{R}$

If a, b and c are all nonzero, L can be described by the symmetric equation $\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$

Examples

Find parametric and symmetric equations of the line L passing through points $P = (3, 2, 1)$ and $Q = (5, 1, -2)$

First, identify the direction vector ($\overrightarrow{PQ}$ or $\overrightarrow{QP}$)

$$\overrightarrow{PQ} = \langle 2, -1, -3 \rangle$$

Take a point on the line (either P or Q).

Parametric equation :
(x, y, z)

$$\begin{cases} x = 3 + 2t \\ y = 2 - t \\ z = 1 - 3t \end{cases}$$

Symmetric equation :

$$\frac{x-3}{2} = \frac{y-2}{-1} = \frac{z-1}{-3}$$