MATH 10C: Calculus III (Lecture B00)

mathweb.ucsd.edu/~ynemish/teaching/10c

Today: The dot product

Next: Strang 2.4

Week 2:

- homework 1 (due Monday, October 3)
- survey on Canvas Quizzes (due Friday, October 7)

Dot product (scalar product) of vectors

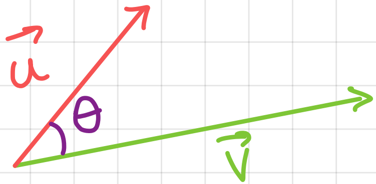
Def If $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ are two vectors in $\mathbb{R}^3$, then the dot product or the scalar product of $\vec{v}$ and $\vec{w}$ is given by the sum of products of vector components

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

(in $\mathbb{R}^2$ $\vec{v} = \langle v_1, v_2 \rangle$
$\vec{u} = \langle u_1, u_2 \rangle$
$\vec{v} \cdot \vec{u} = v_1 u_1 + v_2 u_2$

Theorem 2.4

If



, $0 \le \theta \le \pi$,

then $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta$

Dot product and angles between vectors

From Theorem 2.4 we have

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}, \quad \theta = \arccos\left(\frac{\langle \vec{u}, \vec{v}\rangle}{\|\vec{u}\|\,\|\vec{v}\|}\right)$$

Examples

Find the angle between $\vec{u}$ and $\vec{v}$

(a) $\vec{u} = -\vec{i} + 2\vec{j} - \vec{k}$, $\vec{v} = \vec{i} + 2\vec{j}$

$\|\vec{u}\| =$ $\|\vec{v}\| =$

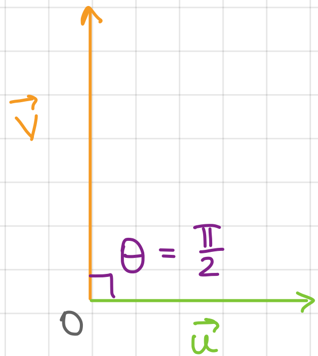
$\vec{u}\cdot\vec{v} =$ $\Rightarrow$ $\cos\theta =$, $\theta =$

(b) $\vec{u} = \langle 1, 2, 3\rangle$, $\vec{v} = \langle -7, 2, 1\rangle$

$\vec{u}\cdot\vec{v} =$ $\Rightarrow$ $\cos\theta =$ $\Rightarrow$ $\theta =$

Orthogonal vectors

If $\cos\theta = 0$, then $\theta = \frac{\pi}{2}$, which means that the vectors form a right angle



We call such vectors

Theorem 2.5

The nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal

Example Determine whether $\vec{p} = \langle 1, 3, 0 \rangle$ and $\vec{q} = \langle -6, 2, 5 \rangle$ are orthogonal. Since $\vec{p} \cdot \vec{q} =$

we conclude that $\vec{p}$ and $\vec{q}$ are

Orthogonality of standard unit vectors

Recall: $\vec{i} = \langle 1,0,0\rangle$, $\vec{j} = \langle 0,1,0\rangle$, $\vec{k} = \langle 0,0,1\rangle$

Then $\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$

$\vec{i}\cdot\vec{j} =$

$\vec{i}\cdot\vec{j} = \vec{i}\cdot\vec{k} = \vec{j}\cdot\vec{k} =$

We say that $\vec{i}, \vec{j}, \vec{k}$ are

Example $(10\vec{i} - \vec{j})\cdot(-\vec{i} + 2\vec{k})$

$=$

$=$

$\langle 10,-1,0\rangle \cdot \langle -1,0,2\rangle =$

Using vectors to represent data

Fruit vendor sells apples, bananas and oranges.
On a given day he sells 30 apples, 12 bananas and 18 oranges. Define the vector

$$\vec{q} =$$

Suppose that the vendor sets the following prices

0.5 per apple, 0.25 per banana, 1 per orange

Define the vector of prices

$$\vec{p} =$$

Then $\vec{q} \cdot \vec{p} =$

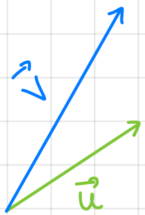
is vendor's

Projections

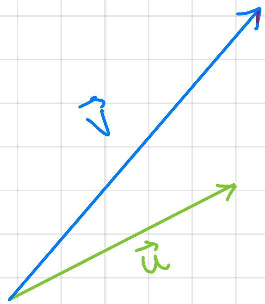
Let $\vec{u}$ and $\vec{v}$ be two vectors. Sometimes we want to decompose $\vec{v}$ into two components

$\vec{v} = \vec{a} + \vec{b}$ such that $\vec{a}$ is parallel to $\vec{u}$

and $\vec{b}$ is orthogonal to $\vec{u}$



Why?

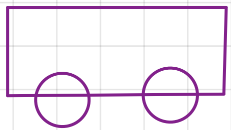


① Find the area of

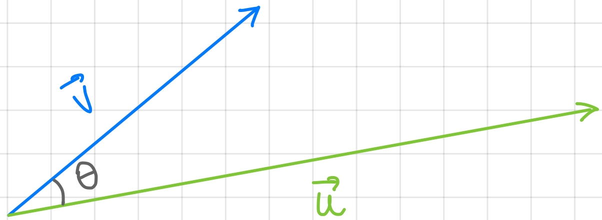
Area of this triangle is

② Child pulls a wagon

How much force is actually moving the wagon forward?



Projections



$\|\vec{a}\| =$

$=$

$\vec{a} =$

$=$

Def (Projection). The vector projection of $\vec{v}$ onto $\vec{u}$ is the vector labeled $\text{proj}_{\vec{u}}\vec{v}$ given by

$$\text{proj}_{\vec{u}}\vec{v} =$$

The length of $\text{proj}_{\vec{u}}\vec{v}$, $\|\text{proj}_{\vec{u}}\vec{v}\| =$

is called the scalar projection of $\vec{v}$ onto $\vec{u}$

Projections

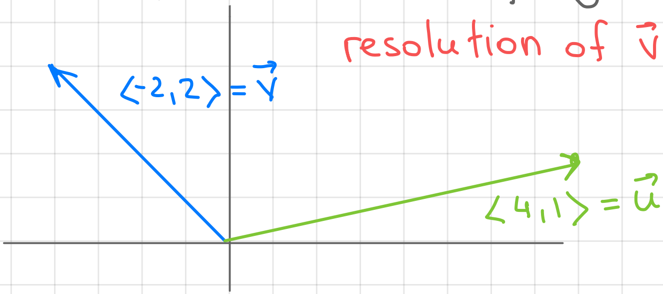
Let $\vec{v}$ and $\vec{u}$ be nonzero vectors. Then

$\vec{u}$ and $\vec{v} - \text{proj}_{\vec{u}}\vec{v}$ are

$$\vec{u} \cdot \left(\vec{v} - \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u} \right) =$$

Example Find the projection of $\langle -2, 2 \rangle$ onto $\langle 4, 1 \rangle$

resolution of $\vec{v}$



$\text{proj}_{\vec{u}}\vec{v} =$

$=$

$=$

$\vec{v} - \text{proj}_{\vec{u}}\vec{v} =$

Projections

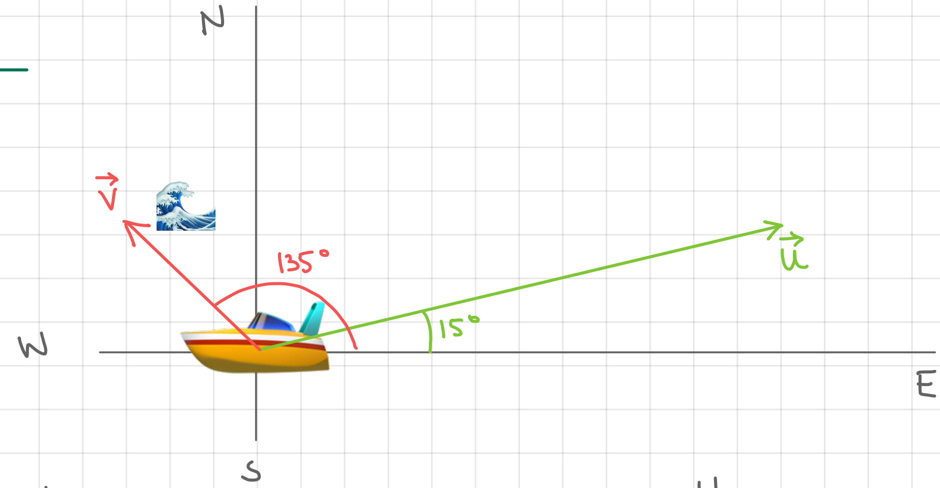
Example Find the projection of $\langle -2, 2, 3\rangle$ onto $\langle 10, -1, 0\rangle$

$$\text{proj}_{\vec{u}}\,\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\,\vec{u}$$

$$=$$

$$=$$

Example



Ship travels 15° north of east with engine generating a speed of 20 knots in this direction. The ocean current moves the ship NW at a speed of 2 knots. How much does the current slow the movement of the ship in the direction of $\vec{u}$.

$\text{proj}_{\vec{u}} \vec{v} =$

The cross product

Def Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$.

Then the cross product of $\vec{u}$ and $\vec{v}$ is vector

$$\vec{u} \times \vec{v} =$$

$$=$$

Example $\vec{p} = \langle 1, 2, 3 \rangle$, $\vec{q} = \langle -1, 2, 0 \rangle$

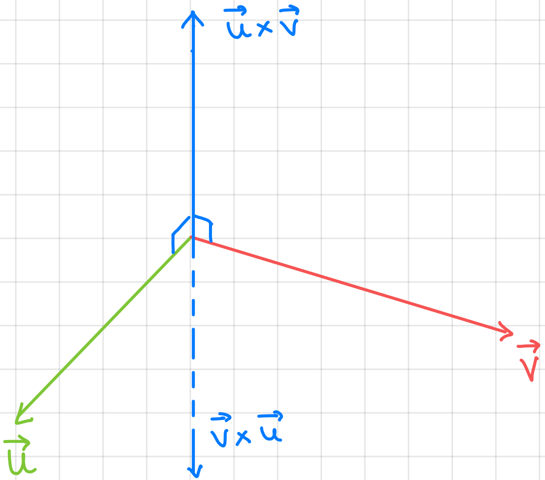
$$\vec{p} \times \vec{q} =$$

$$=$$

$\vec{p} \cdot (\vec{p} \times \vec{q}) =$, $\vec{q} \cdot (\vec{p} \times \vec{q}) =$

The cross product

Fact: Vector $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$!



and the direction is determined by the right-hand rule.

Indeed,

$\vec{p} = \langle 1, 2, 3 \rangle$, $\vec{q} = \langle -1, 2, 0 \rangle$, $\vec{p} \times \vec{q} = \langle -6, -3, 4 \rangle$

$\vec{q} \times \vec{p} =$

$=$

Properties of the cross product

Exercise $\vec{i} \times \vec{j} = \langle 1,0,0\rangle \times \langle 0,1,0\rangle =$

$=$

$\vec{i} \times \vec{j} =$ $\qquad$ $\vec{j} \times \vec{k} =$ $\qquad$ $\vec{k} \times \vec{i} =$

Theorem 2.6 Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$. Then

(i) $\vec{u} \times \vec{v} =$

(ii) $\vec{u} \times (\vec{v} + \vec{w}) =$

(iii) $c(\vec{u} \times \vec{v}) =$

(iv) $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} =$

(v) $\vec{v} \times \vec{v} =$

(vi) $\vec{u} \cdot (\vec{v} \times \vec{w}) =$

For proof expand both sides in terms of components of $\vec{u}, \vec{v}, \vec{w}$