MATH 10C: Calculus III (Lecture B00)

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Today: The dot product

Next: Strang 2.4

Week 2:

- homework 1 (due Monday, October 3)
- survey on Canvas Quizzes (due Friday, October 7)

Dot product (scalar product) of vectors

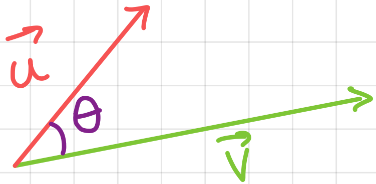
Def If $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ are two vectors in $\mathbb{R}^3$, then the dot product or the scalar product of $\vec{v}$ and $\vec{w}$ is given by the sum of products of vector components

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

(in $\mathbb{R}^2$ $\vec{v} = \langle v_1, v_2 \rangle$
$\vec{u} = \langle u_1, u_2 \rangle$
$\vec{v} \cdot \vec{u} = v_1 u_1 + v_2 u_2$

Theorem 2.4

If



, $0 \le \theta \le \pi$,

then $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta$

Dot product and angles between vectors

From Theorem 2.4 we have

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}, \quad \theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right)$$

Examples

Find the angle between $\vec{u}$ and $\vec{v}$

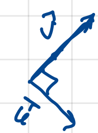
(a) $\vec{u} = -\vec{i} + 2\vec{j} - \vec{k}, \quad \vec{v} = \vec{i} + 2\vec{j}$

$$\|\vec{u}\| = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6} \qquad \|\vec{v}\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\vec{u}\cdot\vec{v} = (-1)\cdot 1 + 2\cdot 2 + (-1)\cdot 0 = 3 \quad \Rightarrow \quad \cos\theta = \frac{3}{\sqrt{6}\cdot\sqrt{5}}, \quad \theta = \arccos\left(\frac{3}{\sqrt{30}}\right)$$

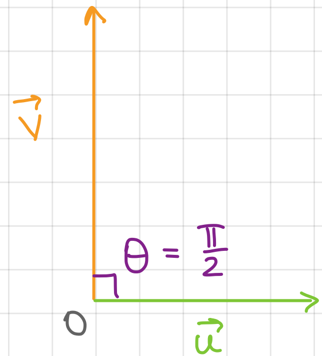
(b) $\vec{u} = \langle 1, 2, 3\rangle, \quad \vec{v} = \langle -7, 2, 1\rangle$

$$\vec{u}\cdot\vec{v} = 1\cdot(-7) + 2\cdot 2 + 3\cdot 1 = 0 \quad \Rightarrow \quad \cos\theta = 0 \quad \Rightarrow \quad \theta = \frac{\pi}{2} = \arccos(0)$$



Orthogonal vectors

If $\cos\theta = 0$, then $\theta = \frac{\pi}{2}$, which means that the vectors form a right angle



We call such vectors

orthogonal (or perpendicular)

Theorem 2.5

The nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u} \cdot \vec{v} = 0$

Example Determine whether $\vec{p} = \langle 1, 3, 0 \rangle$ and $\vec{q} = \langle -6, 2, 5 \rangle$ are orthogonal. Since $\vec{p} \cdot \vec{q} = 1 \cdot (-6) + 3 \cdot 2 + 0 \cdot 5 = 0$ we conclude that $\vec{p}$ and $\vec{q}$ are orthogonal

Orthogonality of standard unit vectors

Recall: $\vec{i} = \langle 1,0,0 \rangle$, $\vec{j} = \langle 0,1,0 \rangle$, $\vec{k} = \langle 0,0,1 \rangle$

Then $\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$

$\vec{i}\cdot\vec{j} = \langle 1,0,0 \rangle \cdot \langle 0,1,0 \rangle = 1\cdot 0 + 0\cdot 1 + 0\cdot 0 = 0$

$\vec{i}\cdot\vec{j} = \vec{i}\cdot\vec{k} = \vec{j}\cdot\vec{k} = 0$

We say that $\vec{i}, \vec{j}, \vec{k}$ are mutually orthogonal

Example $(10\vec{i} - \vec{j})\cdot(-\vec{i} + 2\vec{k})$

$= 10\vec{i}\cdot(-\vec{i} + 2\vec{k}) - \vec{j}\cdot(-\vec{i} + 2\vec{k})$

$= -10\,\underbrace{\vec{i}\cdot\vec{i}}_{=1} + 20\,\underbrace{\vec{i}\cdot\vec{k}}_{=0} + \underbrace{\vec{j}\cdot\vec{i}}_{=0} - 2\,\underbrace{\vec{j}\cdot\vec{k}}_{=0} = -10$

$\langle 10,-1,0 \rangle \cdot \langle -1,0,2 \rangle = -10$

Using vectors to represent data

Fruit vendor sells apples, bananas and oranges. On a given day he sells 30 apples, 12 bananas and 18 oranges. Define the vector

$$\vec{q} = \langle 30, 12, 18 \rangle \quad \text{(quantities)}$$

Suppose that the vendor sets the following prices
0.5 per apple, 0.25 per banana, 1 per orange

Define the vector of prices

$$\vec{p} = \langle 0.5, 0.25, 1 \rangle$$

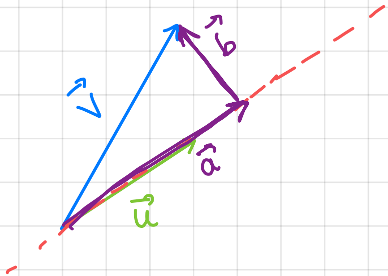
Then $\quad \vec{q} \cdot \vec{p} = 30 \cdot 0.5 + 12 \cdot 0.25 + 18 \cdot 1 = 36$

is vendor's revenue

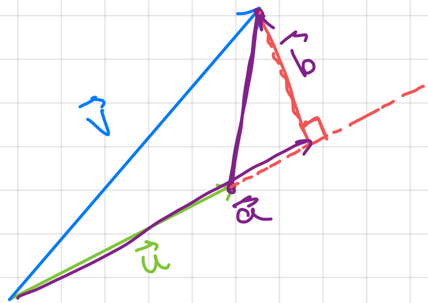
Projections

Let $\vec{u}$ and $\vec{v}$ be two vectors. Sometimes we want to decompose $\vec{v}$ into two components

$\vec{v} = \vec{a} + \vec{b}$ such that $\vec{a}$ is parallel to $\vec{u}$ and $\vec{b}$ is orthogonal to $\vec{u}$



Why?

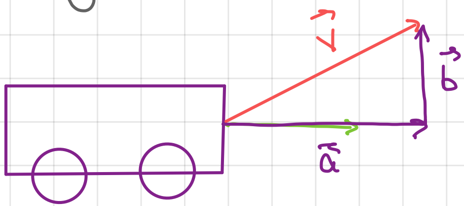


① Find the area of

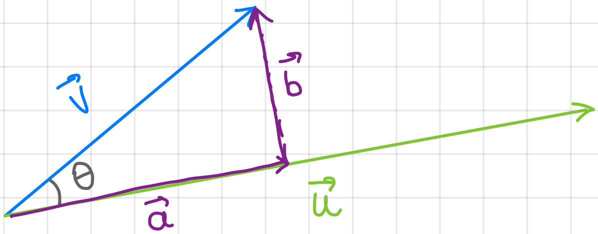
Area of this triangle is

② Child pulls a wagon

How much force is actually moving the wagon forward?



Projections



$$\|\vec{a}\| = \|\vec{v}\| \cdot \cos\theta$$

$$= \|\vec{v}\| \cdot \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}$$

$$\vec{a} = \vec{u} \cdot \frac{1}{\|\vec{u}\|} \cdot \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}$$

$$= \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2} \cdot \vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\,\vec{u}$$

$$\vec{v} = \vec{a} + \vec{b}, \quad \vec{b} = \vec{v} - \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\,\vec{u}$$

<u>Def</u> (Projection). The vector projection of $\vec{v}$ onto $\vec{u}$ is the vector labeled $\text{proj}_{\vec{u}}\vec{v}$ given by

$$\text{proj}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\,\vec{u}$$

The length of $\text{proj}_{\vec{u}}\vec{v}$, $\|\text{proj}_{\vec{u}}\vec{v}\| = \dfrac{|\vec{u}\cdot\vec{v}|}{\|\vec{u}\|^2} \cdot \|\vec{u}\| = \dfrac{|\vec{u}\cdot\vec{v}|}{\|\vec{u}\|}$

is called the scalar projection of $\vec{v}$ onto $\vec{u}$

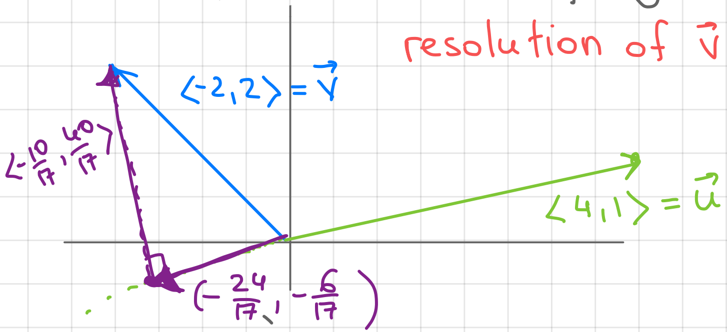
Projections

Let $\vec{v}$ and $\vec{u}$ be nonzero vectors. Then

$\vec{u}$ and $\vec{v} - \text{proj}_{\vec{u}} \vec{v}$ are

$$\vec{u} \cdot \left(\vec{v} - \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u} \right) = \vec{u} \cdot \vec{v} - \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u} \cdot \vec{u} = \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v} = 0$$

Example Find the projection of $\langle -2, 2 \rangle$ onto $\langle 4, 1 \rangle$



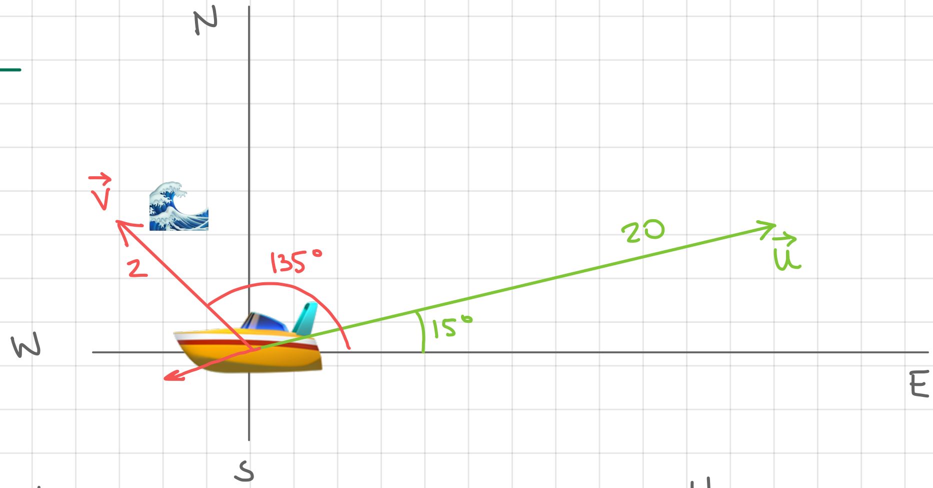
$$\text{proj}_{\vec{u}} \vec{v} = \frac{\langle 4, 1 \rangle \cdot \langle -2, 2 \rangle}{\|\langle 4, 1 \rangle\|^2} \langle 4, 1 \rangle$$

$$= \frac{4 \cdot (-2) + 1 \cdot 2}{17} \langle 4, 1 \rangle$$

$$= \frac{-6}{17} \langle 4, 1 \rangle = \langle -\frac{24}{17}, -\frac{6}{17} \rangle$$

$$\vec{v} - \text{proj}_{\vec{u}} \vec{v} = \langle -\frac{10}{17}, \frac{40}{17} \rangle$$

Example



Ship travels 15° north of east with engine generating a speed of 20 knots in this direction. The ocean current moves the ship NW at a speed of 2 knots. How much does the current slow the movement of the ship in the direction of $\vec{u}$.

$$\text{proj}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u} = \frac{\|\vec{u}\| \, \|\vec{v}\| \cos\theta}{\|\vec{u}\|^2} \vec{u}, \quad \|\text{proj}_{\vec{u}} \vec{v}\| = 1$$