MATH 10C: Calculus III (Lecture B00)

mathweb.ucsd.edu/~ynemish/teaching/10c

Today: Vectors in three dimensions

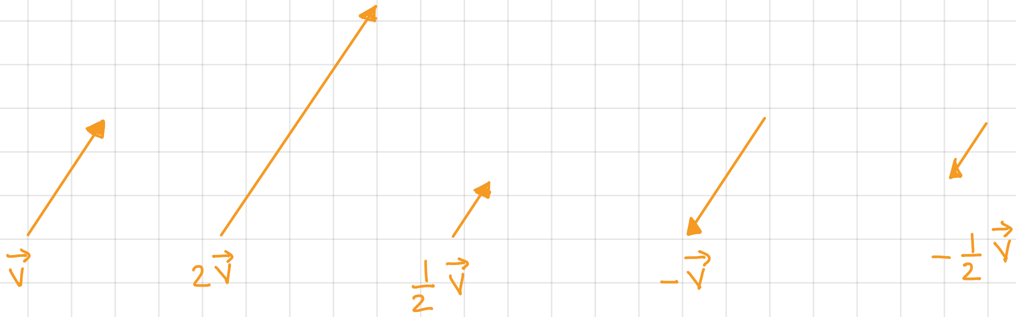
Next: Strang 2.3

Week 1:

- office hours schedule
- homework 1 (due Monday, October 3)
- join Piazza, Edfinity

Last time Vector operations

Scalar multiplication: $k\langle x, y\rangle = \langle kx, ky\rangle$

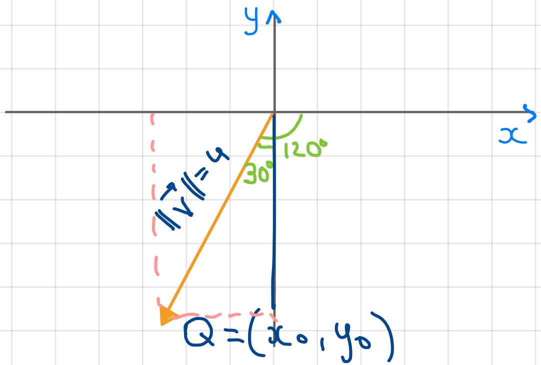


Vector addition: $\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1+x_2, y_1+y_2\rangle$

Vector components and trigonometry

We can describe the direction of the vector in different ways. For example, using the angle that the vector forms with the axes. We can switch between this representation and the component form using trigonometry.

Example Find the component form of a vector with magnitude 4 that forms an angle $-120°$ with the x-axis.



$$|y_0| = 4 \cdot \cos\left(\frac{\pi}{6}\right) = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$$

$$|x_0| = 4 \cdot \sin\left(\frac{\pi}{6}\right) = 4 \cdot \frac{1}{2} = 2$$

$$\vec{v} = \langle -2, -2\sqrt{3} \rangle$$

Unit vectors. Standard unit vectors

A unit vector is a vector with magnitude 1.

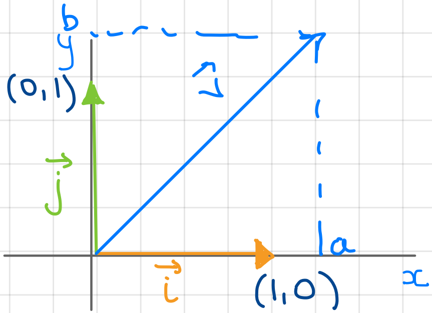
For any nonzero vector $\vec{v}$ we can find a unit vector $\vec{u}$ that has the same direction as $\vec{v}$

Take $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$, then $\vec{u}$ has the same direction as $\vec{v}$

and $\|\vec{u}\| = \left\| \frac{1}{\|\vec{v}\|}\vec{v} \right\| = \frac{1}{\|\vec{v}\|}\cdot\|\vec{v}\| = 1$

Example $\vec{v} = \langle -1, 4\rangle$, $\|\vec{v}\| = \sqrt{1^2+4^2} = \sqrt{17}$, $\vec{u} = \langle -\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\rangle$

Consider the vectors $\vec{i} := \langle 1,0\rangle$, $\vec{j} := \langle 0,1\rangle$



We call $\vec{i}$ and $\vec{j}$ the standard unit vectors in the plane, $\|\vec{i}\| = \|\vec{j}\| = 1$

We can write any vector in the plane as a combination on $\vec{i}$ and $\vec{j}$

$\vec{v} = \langle a,b\rangle$, then $\vec{v} = a\cdot\vec{i} + b\cdot\vec{j} = \langle a,0\rangle + \langle 0,b\rangle$

Vectors in the plane. Summary

- geometrically/physically vectors describe displacement, velocity, force; in plane they represented by arrows
- two vector operations: scalar multiplication, vector addition
- coordinates make vector operations easy to perform
- component form of a vector $\vec{v} = \langle x_1, y_1 \rangle$, $\vec{w} = \langle x_2, y_2 \rangle$
- scalar multiplication and vector addition become componentwise: $k\vec{v} = \langle kx_1, ky_1 \rangle$, $\vec{v} + \vec{w} = \langle x_1 + x_2, y_1 + y_2 \rangle$

$$k_1\vec{v} + k_2\vec{w} = \langle k_1x_1 + k_2x_2,\ k_1y_1 + k_2y_2 \rangle$$

- $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$ are called standard unit vectors
- $\vec{v} = \langle x, y \rangle$ can be written as a combination of $\vec{i}$ and $\vec{j}$

$$\langle x, y \rangle = x\vec{i} + y\vec{j}$$

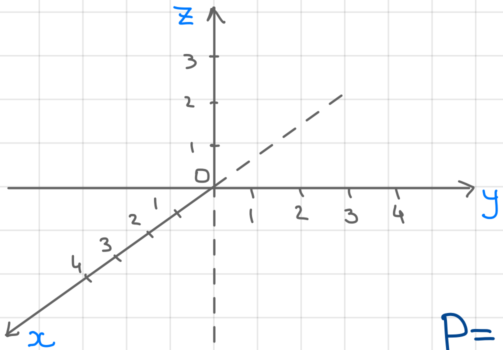
Points in three dimensions

Life happens in three dimensions!

The mathematical model of the three-dimensional space is the three-dimensional rectangular coordinate system $\mathbb{R}^3$.

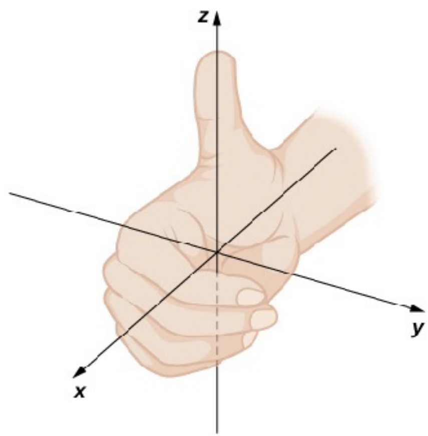
$\mathbb{R}^3$ consists of points (x, y, z), where x, y, z are real numbers

1D: $\mathbb{R}$, 2D: $\mathbb{R}^2$, 3D: $\mathbb{R}^3$



We arrange the axes using the "right hand rule"

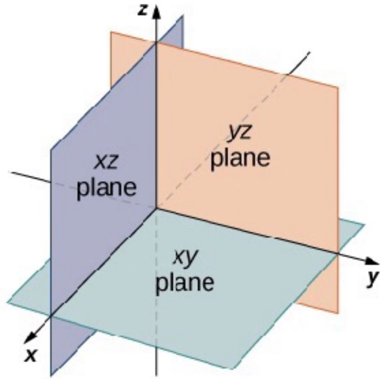
$P = (2, 3, -1)$



Coordinate planes . Octants

There are three axes in $\mathbb{R}^3$ (orthogonal to each other).

If we fix any two axes we get a coordinate plane

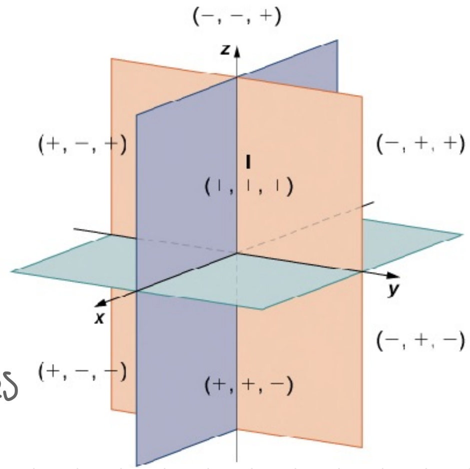


xy plane : $\{(x,y,0) : x,y \in \mathbb{R}\}$ setting $z=0$

xz plane : $\{(x,0,z) : x,z \in \mathbb{R}\}$ setting $y=0$

yz plane : $\{(0,y,z) : y,z \in \mathbb{R}\}$ setting $x=0$

Three coordinate planes split $\mathbb{R}^3$ into eight octants consisting of points with three nonzero coordinates



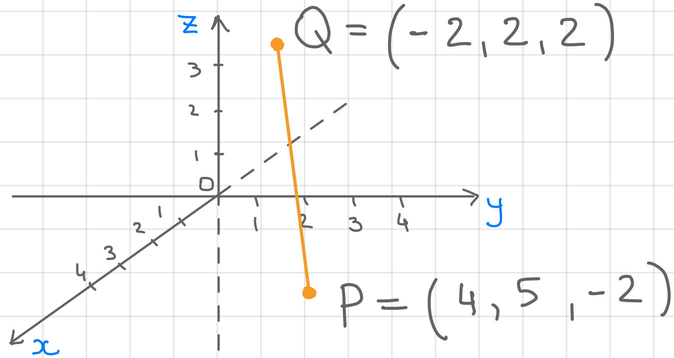
Distance in $\mathbb{R}^3$

Theorem 2.2. Distance between two points in space

The distance d between points $P=(x_1, y_1, z_1)$ and $Q=(x_2, y_2, z_2)$ is given by the formula

$$d(P,Q) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Example

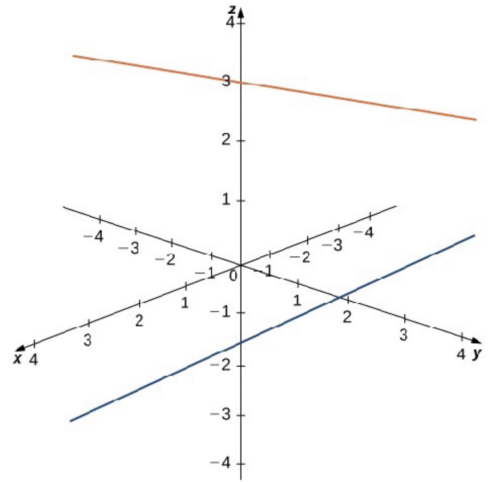


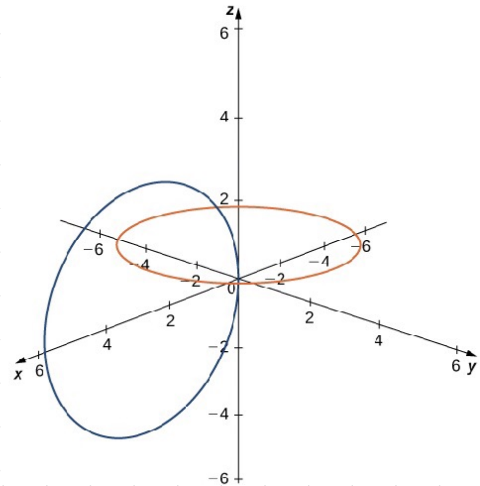
$$d(P,Q) = \sqrt{(-2-4)^2 + (2-5)^2 + (2-(-2))^2}$$

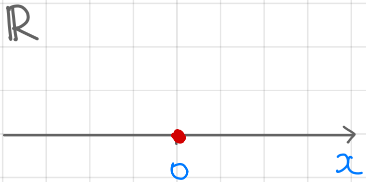
$$= \sqrt{6^2 + 3^2 + 4^2}$$

$$= \sqrt{36 + 9 + 16} = \sqrt{61}$$

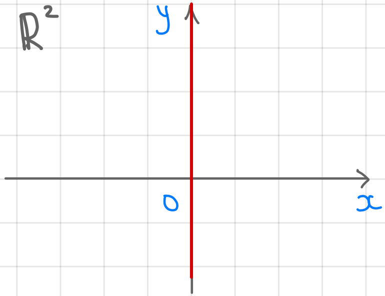
Equations in $\mathbb{R}^3$



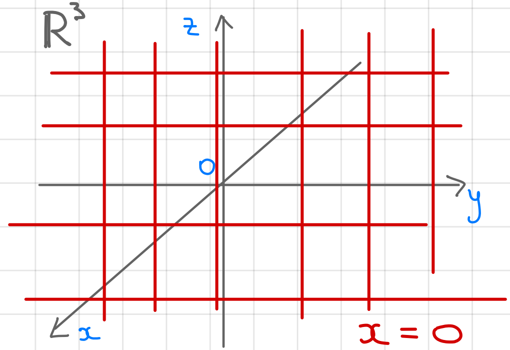




$x = 0$



$x = 0$



$x = 0$

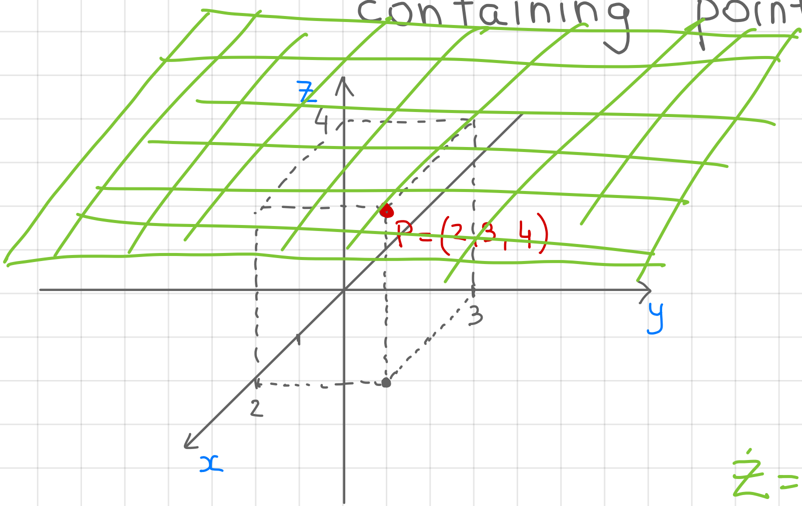
Equations of planes parallel to coordinate planes

Rule

$z=c$: equation of a plane parallel to the xy-plane containing point $P=(a,b,c)$

$y=b$: equation of a plane parallel to the xz-plane containing point $P=(a,b,c)$

$x=a$: equation of a plane parallel to the yz-plane containing point $P=(a,b,c)$



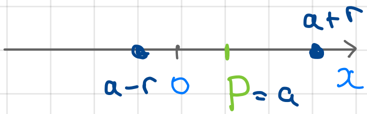
Write an equation of the plane parallel to xy-plane passing through the point $P=(2,3,4)$

$z=4$

Equation of a sphere

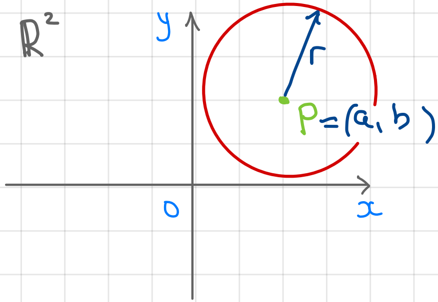
Given point P, describe all points that are at distance $r > 0$ from P.

$\mathbb{R}$



$P = a$, $|x - a| = r$

$\mathbb{R}^2$

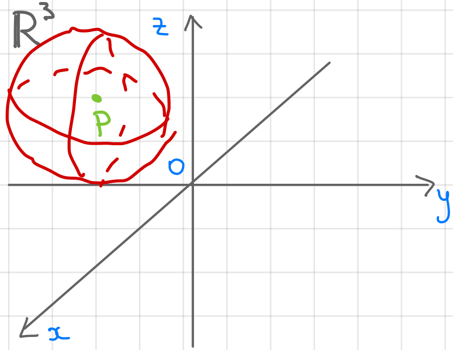


$P = (a, b)$

$(x, y) : \sqrt{(x-a)^2 + (y-b)^2} = r$

$(x-a)^2 + (y-b)^2 = r^2$

$\mathbb{R}^3$



$P = (a, b, c)$

$\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r$

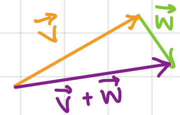
$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$

standard equation of a sphere with center (a, b, c) and radius $r > 0$

Vectors in $\mathbb{R}^3$

Complete analogy with vectors in the plane

- vectors are quantities with both magnitude and direction
- vectors are represented by directed line segments (arrows)
- vector is in the standard position if its initial point is $(0,0,0)$
- vectors admit the component representation $\vec{v} = \langle x, y, z \rangle$
- $\vec{0} = \langle 0, 0, 0 \rangle$
- vector addition and scalar multiplication are defined analogously to plane vectors :



- in the component form :

$$k_1 \langle x_1, y_1, z_1 \rangle + k_2 \langle x_2, y_2, z_2 \rangle = \langle k_1 x_1 + k_2 x_2 ,\ k_1 y_1 + k_2 y_2 ,\ k_1 z_1 + k_2 z_2 \rangle$$

- $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$ are standard unit vectors in $\mathbb{R}^3$

Vectors in $\mathbb{R}^3$

- if $\vec{v} = \langle x, y, z \rangle$, then $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$ (standard unit form)
- if $P = (x_i, y_i, z_i)$, $Q = (x_t, y_t, z_t)$, then $\overrightarrow{PQ} = \langle x_t - x_i, y_t - y_i, z_t - z_i \rangle$
- if $\vec{v} = \langle x, y, z \rangle$, then $\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2}$
- to find the unit vector in the direction $\vec{v} = \langle x, y, z \rangle$, multiply $\vec{v}$ by $\frac{1}{\|\vec{v}\|}$: $\vec{u} = \left\langle \frac{x}{\|\vec{v}\|}, \frac{y}{\|\vec{v}\|}, \frac{z}{\|\vec{v}\|} \right\rangle$

Properties of vector operations (same as in $\mathbb{R}^2$)

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$. Let r, s be scalars.

Then

(i) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutative property)

(ii) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (associative property)

(iii) $\vec{u} + \vec{0} = \vec{u}$ (additive identity property)

(iv) $\vec{u} + (-\vec{u}) = \vec{0}$ (additive inverse property)

(v) $r(s\vec{u}) = (rs)\vec{u}$ (associativity of scalar mult.)

(vi) $(r+s)\vec{u} = r\vec{u} + s\vec{u}$ (distributive property)

(vii) $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$ (distributive property)

(viii) $1 \cdot \vec{u} = \vec{u}$, $0 \cdot \vec{u} = \vec{0}$ (identify and zero properties)