MATH 10C: Calculus III (Lecture B00)

mathweb.ucsd.edu/~ynemish/teaching/10c

Today: Vectors in the plane.
Vectors in three dimensions
Next: Strang 2.3

Week 1:

- check the course website
- homework 1 (due Friday, September 30)
- join Piazza, Edfinity

Last time

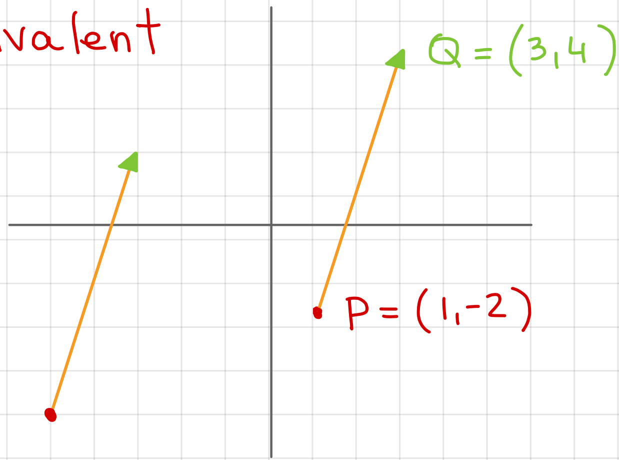
Def. A vector is a quantity that has both magnitude (size, length) and direction

Forces, displacements, velocity are described by vectors.

A vector in a plane is represented by a directed line segment from the initial point to the terminal point.

We say that $\vec{v}$ and $\vec{w}$ are equivalent if they have the same direction and magnitude (denoted $\vec{v} = \vec{w}$).

We treat equivalent vectors as equal.



Scalar multiplication

Let $\vec{v}$ be a vector and k be a real number

Then $k\vec{v}$, called

is a vector such that

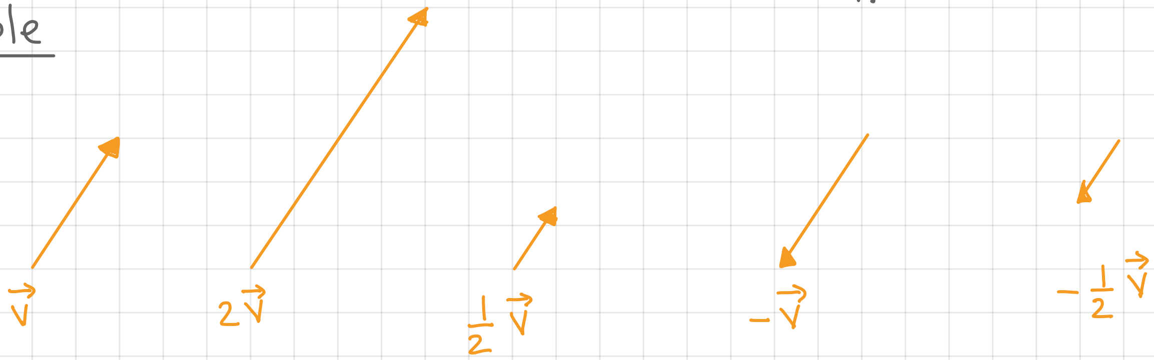
$\|k\vec{v}\| =$

$k\vec{v}$ has the as $\vec{v}$ if

$k\vec{v}$ has the direction to the direction of $\vec{v}$ if

If $k=0$ or $\vec{v}=0$, then

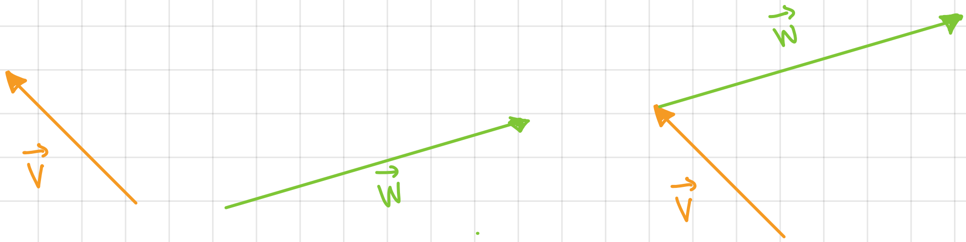
Example



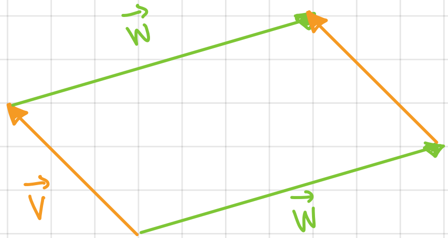
Vector addition

Let $\vec{v}$ and $\vec{w}$ be two vectors. Place the initial point of $\vec{w}$ at the terminal point of $\vec{v}$. Then the vector with initial point at the initial point of $\vec{v}$ and the terminal point at the terminal point of $\vec{w}$ is called the   , and is denoted $\vec{v} + \vec{w}$.

Example



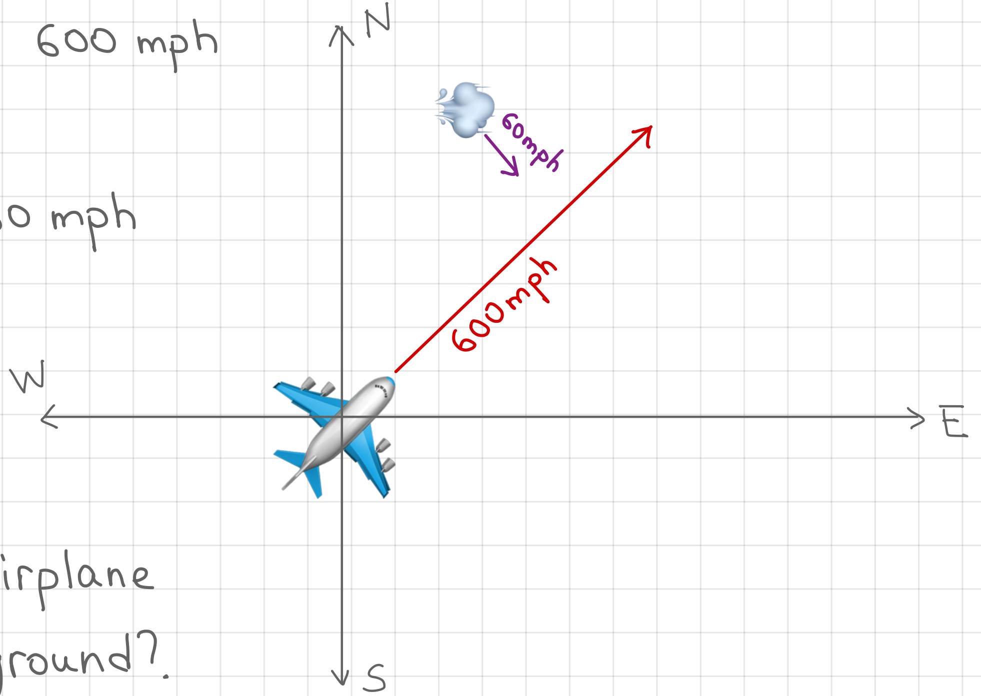
Notice that



Definition of a vector

Airplane flies NE at 600 mph
(relative to the air)

Wind blows SE at 60 mph



How fast does the airplane
fly relative to the ground?
In what direction?

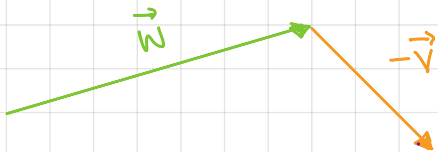
Combining vectors

We know how to define (geometrically) $k_1\vec{v}_1 + k_2\vec{v}_2$

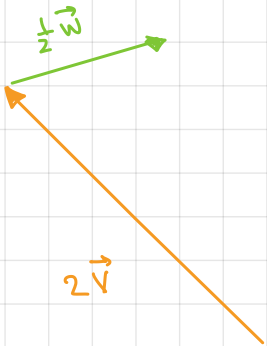
or $k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 \;\ldots$

Example

- $\vec{w} - \vec{v}$

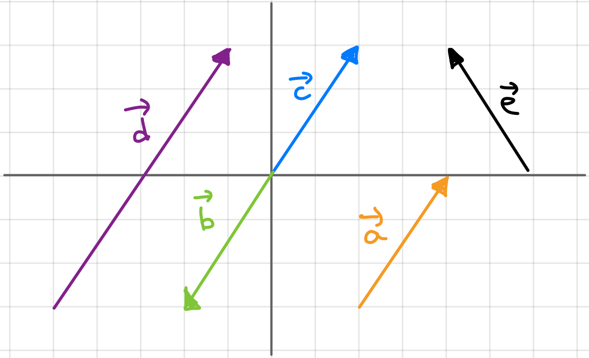


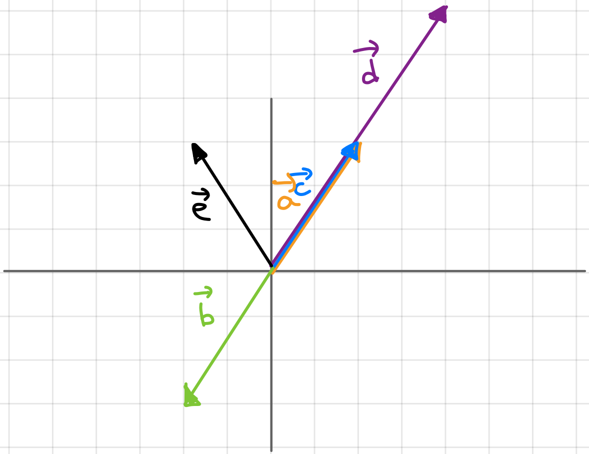
- $2\vec{v} + \frac{1}{2}\vec{w}$



Vector components

It is easier to work with vectors in a coordinate system. Since the location of the initial vector does not matter, let's place all vectors in the plane so that their initial points

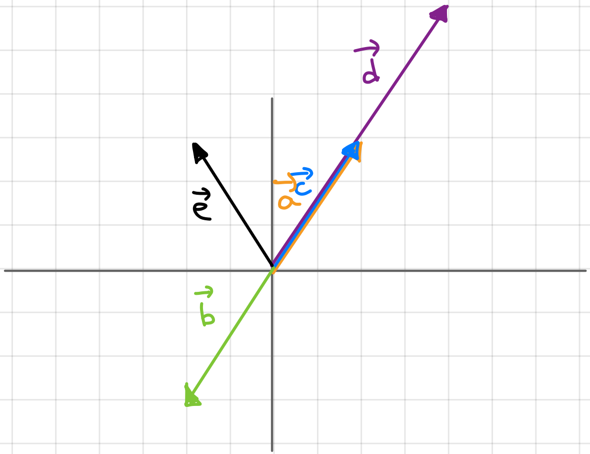




We call such vectors , and they can be described by the

Vector components

Def. The vector with initial point $(0,0)$ and the terminal point (x,y) can be written in component form as   The scalars x and y are called the



$\vec{a} =$

$\vec{b} =$

$\vec{d} =$

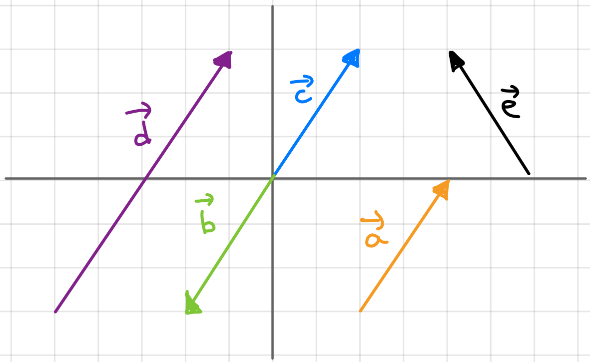
$\vec{e} =$

Vector components

If the vector is not in standard position, but we know the coordinates of its initial and terminal points, then we can find the vectors coordinates using the following rule:

Let $P = (x_i, y_i)$ and $Q = (x_t, y_t)$. Then

$\vec{a}$:



Magnitude of the vector

Magnitude of the vector is the distance between its initial and terminal points.

If $P = (x_i, y_i)$, $Q = (x_t, y_t)$, then

If $\vec{v} = \langle x, y \rangle$, then

Example

- $P = (2, -3)$, $Q = (4, 0)$

- $\vec{a} = \langle 2, 3 \rangle$

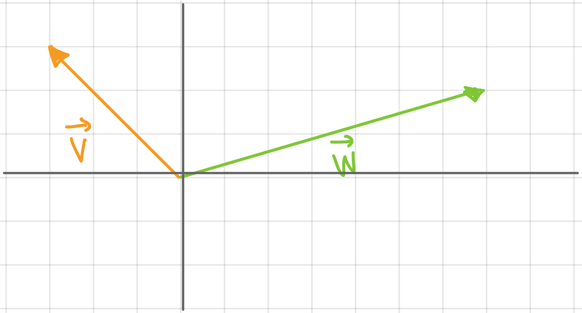
Vector operations in component form

Def. Let $\vec{v} = \langle x_1, y_1 \rangle$, $\vec{w} = \langle x_2, y_2 \rangle$, $k \in \mathbb{R}$.

Then

- $k\vec{v} =$ (scalar multiplication)
- $\vec{v} + \vec{w} =$ (vector addition)

Example



$\vec{v} = \langle -3, 3 \rangle$

$\vec{w} = \langle 7, 2 \rangle$

- $\vec{v} + \vec{w} =$
- $\vec{w} - \vec{v} =$
- $2\vec{v} + \frac{1}{2}\vec{w} =$

Properties of vector operations

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in the plane. Let r, s be scalars.

Then

(i) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutative property)

(ii) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (associative property)

(iii) $\vec{u} + \vec{0} = \vec{u}$ (additive identity property)

(iv) $\vec{u} + (-\vec{u}) = \vec{0}$ (additive inverse property)

(v) $r(s\vec{u}) = (rs)\vec{u}$ (associativity of scalar mult.)

(vi) $(r+s)\vec{u} = r\vec{u} + s\vec{u}$ (distributive property)

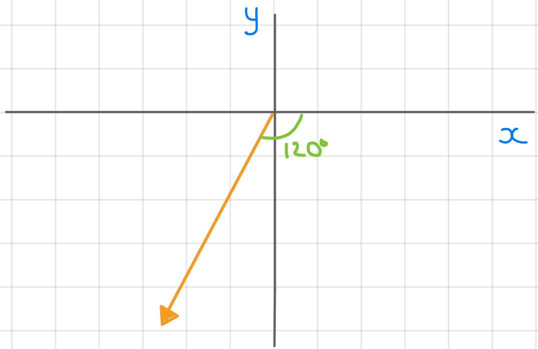
(vii) $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$ (distributive property)

(viii) $1 \cdot \vec{u} = \vec{u}$, $0 \cdot \vec{u} = \vec{0}$ (identify and zero properties)

Vector components and trigonometry

We can describe the direction of the vector in different ways. For example, using the angle that the vector forms with the axes. We can switch between this representation and the component form using trigonometry.

Example Find the component form of a vector with magnitude 4 that forms an angle $-120°$ with the x-axis.



Unit vectors. Standard unit vectors

A unit vector is a vector with magnitude 1.

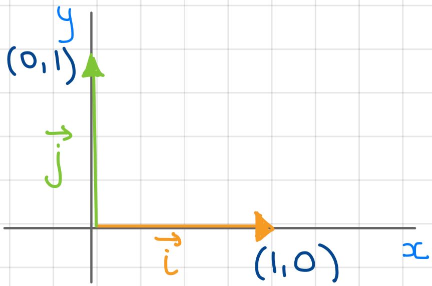
For any nonzero vector $\vec{v}$ we can find a unit vector $\vec{u}$ that has the same direction as $\vec{v}$

Take $\vec{u} =$, then $\vec{u}$ has the same direction as $\vec{v}$

and $\|\vec{u}\| = \| \frac{1}{\|\vec{v}\|} \vec{v} \| =$

<u>Example</u> $\vec{v} = \langle -1, 4 \rangle$, $\|\vec{v}\| =$

Consider the vectors $\vec{i} := \langle 1, 0 \rangle$, $\vec{j} := \langle 0, 1 \rangle$



We call $\vec{i}$ and $\vec{j}$ the

in the plane

We can write any vector in the plane as a combination on $\vec{i}$ and $\vec{j}$

$\vec{v} = \langle a, b \rangle$, then $\vec{v} =$

Vectors in the plane. Summary

- geometrically/physically vectors describe displacement, velocity, force; in plane they represented by arrows
- two vector operations: scalar product and vector sum
- coordinates make vector operations easy to perform
- component form of a vector $\vec{v} = \langle x_1, y_1 \rangle$, $\vec{w} = \langle x_2, y_2 \rangle$
- scalar multiplication and vector addition become componentwise: $k\vec{v} = \langle kx_1, ky_1 \rangle$, $\vec{v} + \vec{w} = \langle x_1 + x_2, y_1 + y_2 \rangle$

$$k_1\vec{v} + k_2\vec{w} = \langle k_1x_1 + k_2x_2, k_1y_1 + k_2y_2 \rangle$$

- $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$ are called standard unit vectors
- $\vec{v} = \langle x, y \rangle$ can be written as a combination of $\vec{i}$ and $\vec{j}$

$$\langle x, y \rangle = x\vec{i} + y\vec{j}$$

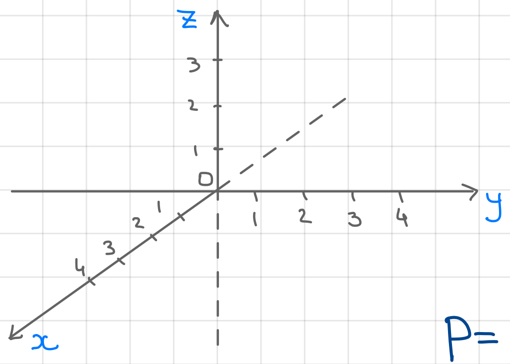
Points in three dimensions

Life happens in three dimensions!

The mathematical model of the three-dimensional space is the three-dimensional rectangular coordinate system $\mathbb{R}^3$.

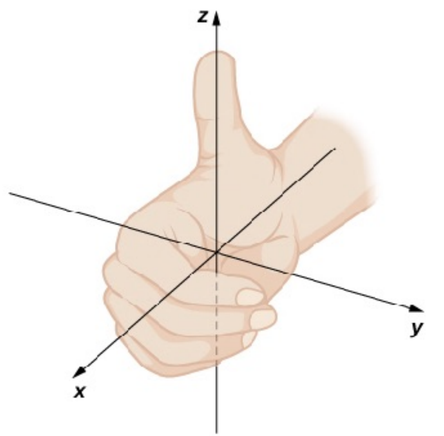
$\mathbb{R}^3$ consists of points (x, y, z), where x, y, z are real numbers

1D: $\mathbb{R}$, 2D: $\mathbb{R}^2$, 3D: $\mathbb{R}^3$



We arrange the axes using the "right hand rule"

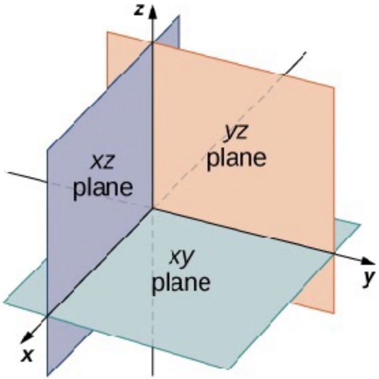
$P = (2, 3, -1)$



Coordinate planes. Octants

There are three axes in $\mathbb{R}^3$ (orthogonal to each other).

If we fix any two axes we get a coordinate plane

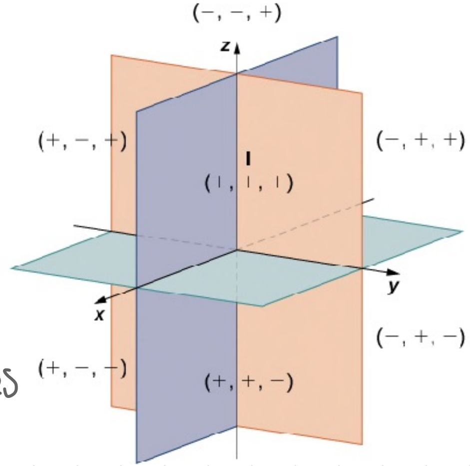


xy plane : $\{\qquad\qquad\qquad\}$ setting

xz plane : $\{\qquad\qquad\qquad\}$ setting

yz plane : $\{\qquad\qquad\qquad\}$ setting

Three coordinate planes split $\mathbb{R}^3$ into eight octants consisting of points with three nonzero coordinates

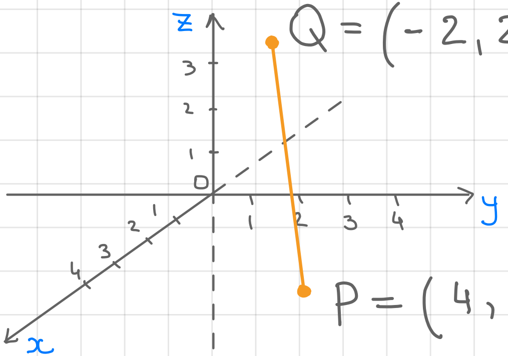


Distance in $\mathbb{R}^3$

Theorem 2.2. Distance between two points in space

The distance d between points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ is given by the formula

Example



$Q = (-2, 2, 2)$

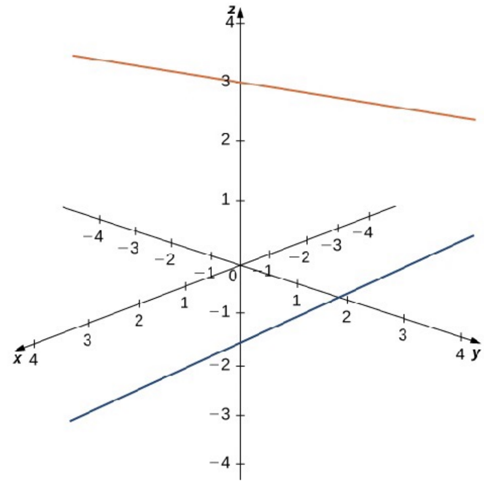
$P = (4, 5, -2)$

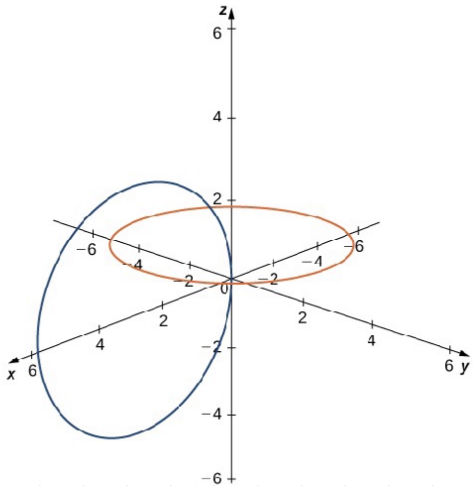
$d(P, Q) =$

$=$

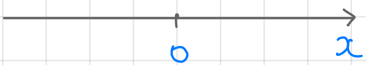
$=$

Equations in $\mathbb{R}^3$



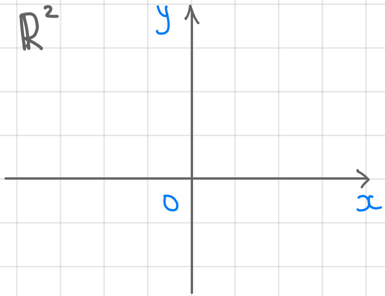


$\mathbb{R}$



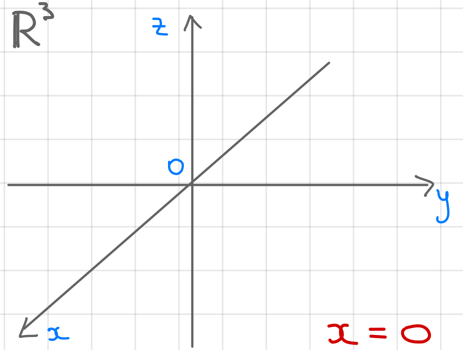
$x = 0$

$\mathbb{R}^2$



$x = 0$

$\mathbb{R}^3$



$x = 0$

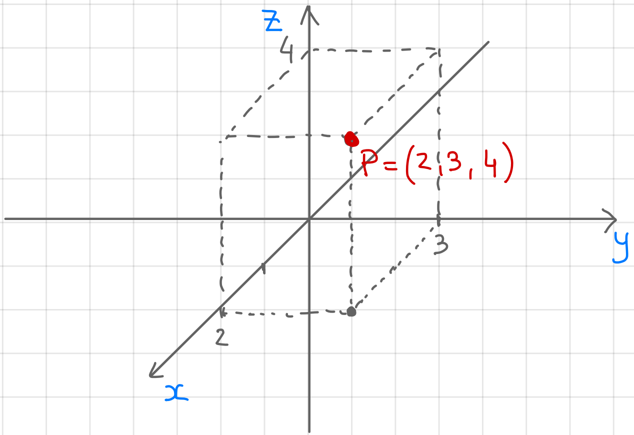
Equations of planes parallel to coordinate planes

Rule

$z = c$: equation of a plane parallel to the xy-plane containing point $P = (a, b, c)$

$y = b$: equation of a plane parallel to the xz-plane containing point $P = (a, b, c)$

$x = a$: equation of a plane parallel to the yz-plane containing point $P = (a, b, c)$

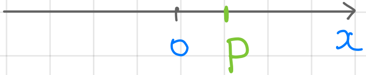


Write an equation of the plane parallel to xy-plane passing through the point $P = (2, 3, 4)$

Equation of a sphere

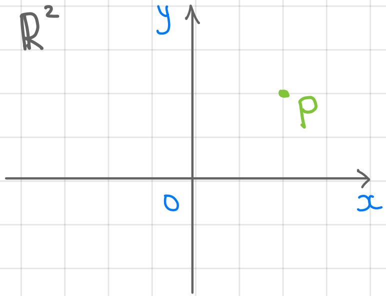
Given point P, describe all points that are at distance $r>0$ from P.

$\mathbb{R}$



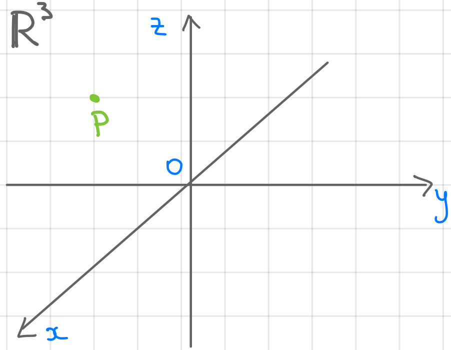
$P = a$,

$\mathbb{R}^2$



$P = (a, b)$

$\mathbb{R}^3$



$P = (a, b, c)$

Equation of a sphere

<u>Example</u> Find the standard equation of the sphere with center $(2,3,4)$ and point $(0,11,-1)$

In order to write the equation of a sphere we need to know the center (given) and the radius (unknown). Radius is the distance from the center of the sphere to any point of the sphere (in particular to $(0,11,-1)$)

Therefore,

$r =$

$=$

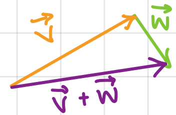
$=$

Equation of the sphere :

Vectors in $\mathbb{R}^3$

Complete analogy with vectors in the plane

- vectors are quantities with both magnitude and direction
- vectors are represented by directed line segments (arrows)
- vector is in the standard position if its initial point is $(0,0,0)$
- vectors admit the component representation $\vec{v} = \langle x, y, z \rangle$
- $\vec{0} = \langle 0, 0, 0 \rangle$
- vector addition and scalar multiplication are defined analogously to plane vectors:



- in the component form:

$$k_1 \langle x_1, y_1, z_1 \rangle + k_2 \langle x_2, y_2, z_2 \rangle = \langle k_1 x_1 + k_2 x_2, k_1 y_1 + k_2 y_2, k_1 z_1 + k_2 z_2 \rangle$$

- $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$ are standard unit vectors in $\mathbb{R}^3$

Vectors in $\mathbb{R}^3$

- if $\vec{v} = \langle x, y, z \rangle$, then $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$ (standard unit form)
- if $P = (x_i, y_i, z_i)$, $Q = (x_t, y_t, z_t)$, then $\overrightarrow{PQ} = \langle x_t - x_i, y_t - y_i, z_t - z_i \rangle$
- if $\vec{v} = \langle x, y, z \rangle$, then $\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2}$
- to find the unit vector in the direction $\vec{v} = \langle x, y, z \rangle$, multiply $\vec{v}$ by $\frac{1}{\|\vec{v}\|}$: $\vec{u} = \left\langle \frac{x}{\|\vec{v}\|}, \frac{y}{\|\vec{v}\|}, \frac{z}{\|\vec{v}\|} \right\rangle$

<u>Example</u> Let $P = (0, 3, -2)$, $Q = (2, 2, 2)$. Express $\overrightarrow{PQ}$ in component form and in standard unit form.

$\overrightarrow{PQ} =$

<u>Example</u> Let $\vec{v} = \langle 2, 0, 6 \rangle$, $\vec{w} = \langle 1, -1, -2 \rangle$. Then

$\vec{v} + 3\vec{w} =$

$\|\vec{v} + 3\vec{w}\| =$

Properties of vector operations

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$. Let r, s be scalars.

Then

(i) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutative property)

(ii) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (associative property)

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