MATH 10C: Calculus III (Lecture B00)

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Today: Method of Lagrange multipliers

Next: Review

Week 10:

- Homework 8 due Friday, December 2
- CAPES 65%
- Final: Monday, December 5, 11:30 AM - 2:30 PM

Method of Lagrange multipliers. One constraint

Problem: find the maximum/minimum of $f(x,y)$ on the curve C that is defined by the equation $g(x,y)=0$. Suppose that f is differentiable and C is smooth.

Problem solving strategy:

2. Set up the system of equations using the following template

$$\begin{cases} \nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0) \\ g(x_0,y_0) = 0 \end{cases}$$

3. Solve for x_0 and y_0 (may have multiple solutions)

4. The largest of the values of f at points (x_0,y_0) found above maximizes f on C; the smallest of the values minimizes f on C.

More about step 4

Lagrange multipliers are used to find the critical points.

The points of local minima/maxima are critical points, but critical points are not necessarily local minima/maxima

Suppose $(x_0,y_0),\dots,(x_n,y_n)$ are the points that satisfy the Lagrange multipliers equation and $f(x_0,y_0) < f(x_1,y_1) \le \cdots < f(x_n,y_n)$

- if $g(x,y)=0$ is bounded, then (x_0,y_0) minimizes f on $g(x,y)=0$, (x_n,y_n) maximizes f on $g(x,y)=0$ (we know max/min exist)
- if $g(x,y)=0$ is unbounded, visualize and determine whether f gets larger or smaller as (x,y) goes to infinity along $g(x,y)=0$
- if $g(x,y)=0$ is unbounded but we consider only a bounded part D of it, then check the value of f at the endpoints (boundary) of D

Method of Lagrange multipliers. Cobb-Douglas function

Company's production level is given by the Cobb-Douglas formula $f(x,y) = 2.5x^{0.45}y^{0.55}$, where x is the total number of labor hours, and y represents the total capital input. Suppose 1 unit of labor costs 40 \$, one unit of capital costs 50 \$. Use the Lagrange multipliers method to find the max value of $f(x,y) = 2.5x^{0.45}y^{0.55}$ subject to budgetary constraint of 500000 \$.

1. Objective function is f, the constraint $40 \cdot x + 50 \cdot y = 500000$

   $g(x,y) = 40x + 50y - 500000$, $x \geq 0$, $y \geq 0$

2. Set up the system of equations:

$$f_x = \underbrace{2.5 \cdot 0.45}_{\frac{9}{8}} \cdot x^{-0.55} y^{0.55}, \quad f_y = \underbrace{2.5 \cdot 0.55}_{\frac{11}{8}} \cdot x^{0.45} y^{-0.45}, \quad g_x = 40, \quad g_y = 50$$

Method of Lagrange multipliers. Cobb-Douglas function

$$(*)\quad \begin{cases} f_x = \dfrac{9}{8}\left(\dfrac{y}{x}\right)^{0.55} = \lambda \cdot 40 = \lambda \cdot g_x & (1) \\[2ex] \dfrac{11}{8}\left(\dfrac{x}{y}\right)^{0.45} = \lambda \cdot 50 & (2) \\[2ex] 40x + 50y = 500000 & (3) \end{cases}$$

3. Solve the system (*):

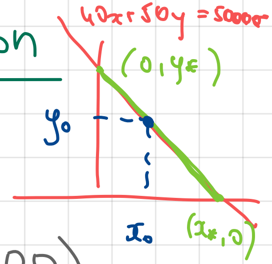
$$(1) \to \quad \lambda = \frac{9}{40\cdot 8}\left(\frac{y}{x}\right)^{0.55} \qquad\qquad (2) \to \quad \frac{11}{8\cdot 50}\left(\frac{x}{y}\right)^{0.45} = \lambda$$

$$\frac{9}{40\cdot 8}\left(\frac{y}{x}\right)^{0.55} = \frac{11}{50\cdot 8}\left(\frac{x}{y}\right)^{0.45} \to \quad \frac{y}{x} = \frac{4\cancel{0}\cdot\cancel{8}}{9}\cdot\frac{11}{5\cancel{0}\cdot\cancel{8}} = \frac{44}{45}$$

$$y = \frac{44}{45}x$$

$$x = 5625, \quad y = 5500$$

Method of Lagrange multipliers. Cobb-Douglas function



$$x = \frac{45000}{8} = 5625 \qquad y = \frac{44000}{8} = 5500$$

4. The candidate for the maximum is $(5625, 5500)$.

Is this a maximum or a minimum?

Consider the function $2.5x^{0.45}y^{0.55}$ on the budgetary constraint line $40x + 50y = 500000$.

f can only have either one max on this line or one min on this line. Compute the value of f at the endpoints: $x=0, y=10000$ and $y=0, x=12500$

$$f(0,10000)=0, \quad f(12500,0)=0 < f(5625,5500) = 2.5\cdot(5625)^{0.45}(5500)^{0.55}$$

Therefore, the production is maximized with 5625 units of labor and 5500 units of capital.

Lagrange multipliers in $\mathbb{R}^3$. One constraint function

Company's production level is given by the Cobb-Douglas function $f(x,y,z) = x^{0.2} y^{0.4} z^{0.4}$, where x is labor, y is capital, z is advertizing. Suppose 1 unit of labor costs 40 \$, one unit of capital costs 50 \$, one unit of advertizing costs 100 \$. Use the Lagrange multipliers method to find the max value of $f(x,y,z)$ subject to budgetary constraints of 500000 \$.

1. Objective function: $f(x,y,z) = x^{0.2} y^{0.4} z^{0.4}$ $\quad \left(\begin{array}{l} x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{array} \right)$

   Constraint function: $g(x,y,z) = 40 \cdot x + 50 \cdot y + 100 \cdot z - 500000$

2. Compute ∇f and ∇g and set up the equations:

$$f_x = 0.2\, x^{-0.8} y^{0.4} z^{0.4} \qquad f_y = 0.4\, x^{0.2} y^{-0.6} z^{0.4} \qquad f_z = 0.4\, x^{0.2} y^{0.4} z^{-0.6}$$

Lagrange multipliers in $\mathbb{R}^3$. One constraint function

Objective function: $f(x,y,z) = x^{0.2}y^{0.4}z^{0.4}$

Constraint function: $g(x,y,z) = 40x + 50y + 100z - 500000 = 0$

$g_x = 40, \quad g_y = 50, \quad g_z = 100$

Equations:
$$\begin{cases} 0.2\, x^{-0.8} y^{0.4} z^{0.4} = \lambda \cdot 40 & (1) \\ 0.4\, x^{0.2} y^{-0.6} z^{0.4} = \lambda \cdot 50 & (2) \\ 0.4\, x^{0.2} y^{0.4} z^{-0.6} = \lambda \cdot 100 & (3) \\ 40x + 50y + 100z = 500000 & (4) \end{cases}$$

3. Solve the system

$$\frac{0.2}{40} x^{-0.8} y^{0.4} z^{0.4} = \lambda$$

$$\frac{0.4}{50} x^{0.2} y^{-0.6} z^{0.4} = \lambda$$

$$\frac{0.4}{100} x^{0.2} y^{0.4} z^{-0.6} = \lambda$$

$$\frac{0.2}{40} x^{-0.8} y^{0.4} \cancel{z^{0.4}} = \frac{0.4}{50} x^{0.2} y^{-0.6} \cancel{z^{0.4}}$$

$$\frac{0.2}{40} x^{-0.8} \cancel{y^{0.4}} z^{0.4} = \frac{0.4}{100} x^{0.2} \cancel{y^{0.4}} z^{-0.6}$$

$$y = \frac{0.4}{50} \cdot \frac{40}{0.2} x, \quad z = \frac{0.4}{100} \frac{40}{0.2} x$$

Lagrange multipliers in $\mathbb{R}^3$. One constraint function

Plug $y = \frac{40}{0.2} \cdot \frac{0.4}{50} x$, $z = \frac{40}{0.2} \cdot \frac{0.4}{100} x$ into (4)

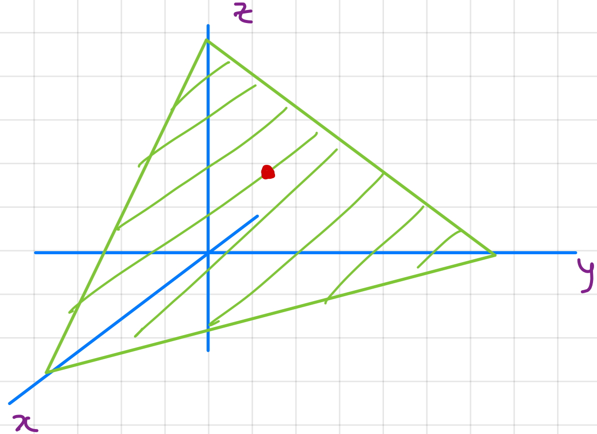
$$40 \cdot x + \cancel{50} \cdot \frac{40}{0.2} \cdot \frac{0.4}{\cancel{50}} x + \cancel{100} \cdot \frac{40}{0.2} \cdot \frac{0.4}{\cancel{100}} x = 500000$$

$$40 \cdot x + 80x + 80x = 500000$$

$$200x = 500000, \quad x = 2500$$

$$y = \frac{80}{50} \cdot 2500 = 4000$$

$$z = \frac{80}{100} \cdot 2500 = 2000$$



4. The candidate for the maximum is $(2500, 4000, 2000)$

Min/max/saddle point? Only one critical point inside

On the boundary ($x=0$ or $y=0$ or $z=0$) $f = 0 < f(2500, 4000, 2000)$

Therefore, $(2500, 4000, 2000)$ is the point of maximum.