MATH 10C: Calculus III (Lecture B00)

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Today: Method of Lagrange multipliers

Next: Strang 4.8

Week 9:

- Homework 7 due Wednesday, November 23
- Homework 8 due Friday, December 2

Method of Lagrange multipliers. One constraint

Problem: find the maximum/minimum of $f(x,y)$ on the curve C that is defined by the equation $g(x,y)=0$. Suppose that f is differentiable and C is smooth.

Problem solving strategy:

2. Set up the system of equations using the following template

$$\begin{cases} \nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0) \\ g(x_0,y_0)=0 \end{cases}$$

3. Solve for x_0 and y_0 (may have multiple solutions)

4. The largest of the values of f at points (x_0,y_0) found above maximizes f on C; the smallest of the values minimizes f on C.

More about step 4

Lagrange multipliers are used to find the critical points.

The points of local minima/maxima are critical points,

but critical points are not necessarily local minima/maxima

Suppose $(x_0,y_0),\dots,(x_n,y_n)$ are the points that satisfy the Lagrange multipliers equation and $f(x_0,y_0) < f(x_1,y_1) \le \dots < f(x_n,y_n)$

- if $g(x,y)=0$ is bounded, then (x_0,y_0) minimizes f on $g(x,y)=0$, (x_n,y_n) maximizes f on $g(x,y)=0$ (we know max/min exist)
- if $g(x,y)=0$ is unbounded, visualize and determine whether f gets larger or smaller as (x,y) goes to infinity along $g(x,y)=0$
- if $g(x,y)=0$ is unbounded but we consider only a bounded part D of it, then check the value of f at the endpoints (boundary) of D

Step 4 of the last example

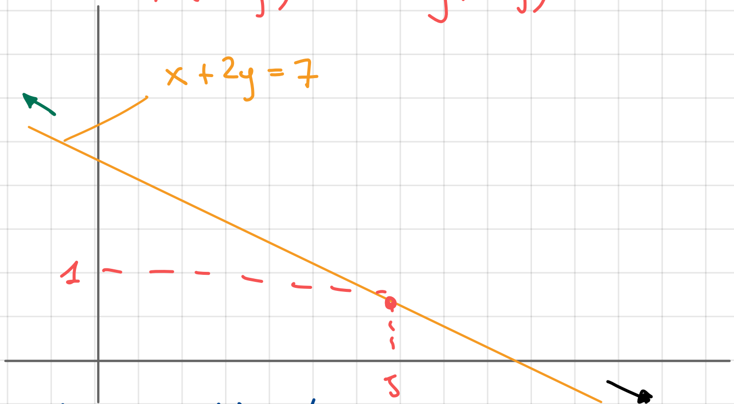
Example Use the method of Largange multipliers to find the minimum value of $f(x,y) = x^2+4y^2-2x+8y$ subject to the constraint $x+2y=7$.

Steps 1-3: Point $(5,1)$ is the only solution to the Lagrange multipliers equation.

Step 4: What is the minimum of $f(x,y)$ on $g(x,y)=0$?

Is $x+2y=7$ bounded? NO

If (x,y) goes to infinity along $g(x,y)=0$ in any direction $|x|$ and $|y|$ become infinitely large,



so f tends to $+\infty$ in ↖ direction and in ↘ direction, $(5,1)$ is a unique critical point, $f(5,1)=27$. Conslusion: $f(5,1)=27$ is the minimum

Method of Lagrange multipliers

Example Maximize $f(x,y) = x(x^2+2y^2-1)$ subject to $x^2+y^2=1$.

1. Maximize $f(x,y) = x(x^2+2y^2-1)$ subject to $g(x,y) = x^2+y^2-1$

2. Set up the system of equations

$$f_x = 3x^2+2y^2-1, \quad g_x = 2x$$

$$f_y = 4xy, \quad g_y = 2y$$

$$\begin{cases} 3x^2+2y^2-1 = \lambda \cdot 2x & (1) \\ 4xy = \lambda 2y & (2) \\ x^2+y^2=1 & (3) \end{cases}$$

3. From (2): either $y=0$ or (divide by y) $\lambda = 2x$

If $y=0$: from (3) $x = \pm 1$

If $y \neq 0$, $\lambda = 2x$: use (3) in (1) $x^2+2\overbrace{(x^2+y^2)}^{=1}-1 = \lambda \cdot 2x$, $x^2+1 = 2\lambda x$

then use $\lambda = 2x$: $x^2+1 = 2\cdot(2x)\cdot x$, $x^2+1 = 4x^2$, $x^2 = \frac{1}{3}$, $x = \pm\sqrt{\frac{1}{3}}$

From (3) $\frac{1}{3} + y^2 = 1$, $y^2 = \frac{2}{3}$, $y = \pm\sqrt{\frac{2}{3}}$

4. Compute $f(1,0)$, $f(-1,0)$, $f\left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right)$, $f\left(\frac{1}{\sqrt{3}}, -\sqrt{\frac{2}{3}}\right)$, $f\left(-\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right)$, $f\left(-\frac{1}{\sqrt{3}}, -\sqrt{\frac{2}{3}}\right)$

Method of Lagrange multipliers. Cobb-Douglas function

Company's production level is given by the Cobb-Douglas formula $f(x,y) = 2.5x^{0.45}y^{0.55}$, where x is the total number of labor hours, and y represents the total capital input. Suppose 1 unit of labor costs 40 \$, one unit of capital costs 50 \$. Use the Lagrange multipliers method to find the max value of $f(x,y) = 2.5x^{0.45}y^{0.55}$ subject to budgetary constraint of 500000 \$.

1. Objective function is f, the constraint $40 \cdot x + 50 \cdot y = 500000$

$$g(x,y) = 40x + 50y - 500000, \quad x \geq 0, y \geq 0$$

2. Set up the system of equations:

$$f_x = \underbrace{2.5 \cdot 0.45}_{\frac{9}{8}} \cdot x^{-0.55} y^{0.55}, \quad f_y = \underbrace{2.5 \cdot 0.55}_{\frac{11}{8}} \cdot x^{0.45} y^{-0.45}, \quad g_x = 40, \quad g_y = 50$$

Method of Lagrange multipliers. Cobb-Douglas function

$$(*)\begin{cases} f_x = \frac{9}{8}\left(\frac{y}{x}\right)^{0.55} = \lambda \cdot 40 = \lambda \cdot g_x & (1) \\ \frac{11}{8}\left(\frac{x}{y}\right)^{0.45} = \lambda \cdot 50 & (2) \\ 40x + 50y = 500000 & (3) \end{cases}$$

3. Solve the system $(*)$:

$$(1) \rightarrow \quad \lambda = \frac{9}{40 \cdot 8}\left(\frac{y}{x}\right)^{0.55} \qquad (2) \rightarrow \quad \frac{11}{8 \cdot 50}\left(\frac{x}{y}\right)^{0.45} = \lambda$$

$$\frac{9}{40 \cdot 8}\left(\frac{y}{x}\right)^{0.55} = \frac{11}{50 \cdot 8}\left(\frac{x}{y}\right)^{0.45} \rightarrow \quad \frac{y}{x} = \frac{4\cancel{0} \cdot \cancel{8}}{9} \cdot \frac{11}{5\cancel{0} \cdot \cancel{8}} = \frac{44}{45}$$

$$y = \frac{44}{45}x$$

$$x = 5625, \quad y = 5500$$