MATH 10C: Calculus III (Lecture B00)

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Today: Directional derivative. Gradient

Next: Strang 4.7

Week 7:

- homework 6 (due Friday, November 11)

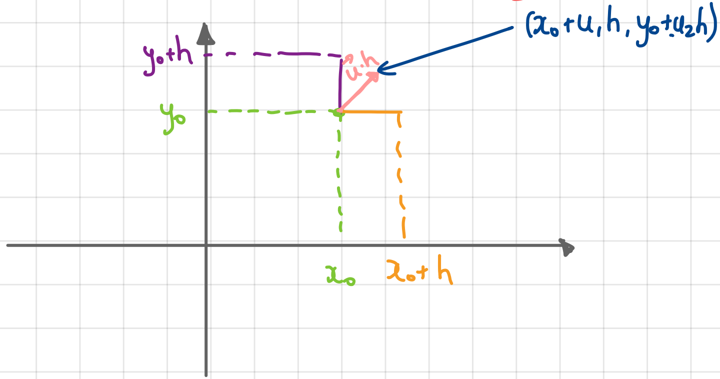
Directional derivative

Consider a function of two variables $f(x,y)$.

Then the partial derivatives $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ represent the rate of change of function f at point (x_0,y_0) in the x-direction and in the y-direction correspondingly.

Q: What if we want to know the rate of change in another direction?

Represent the direction by a unit vector $\vec{u} = \langle u_1, u_2 \rangle$ such that $\sqrt{u_1^2 + u_2^2} = 1$



Want to know the rate of chang in the direction $\vec{u}$.

Directional derivative

$$\vec{u}_x = \langle 1, 0 \rangle$$
$$\vec{u}_y = \langle 0, 1 \rangle$$

Definition We call

$$D_{\vec{u}} f(x_0, y_0) := \lim_{h \to 0} \frac{f(x_0 + h \cdot u_1, y_0 + h \cdot u_2) - f(x_0, y_0)}{h}$$

the directional derlvative of f at poin (x_0, y_0) in the direction $\vec{u} = \langle u_1, u_2 \rangle$ (provided that the limit exists)

Q: How to compute $D_{\vec{u}} f(x_0, y_0)$?

- use the definition; or
- use the following fact: if f is differentiable, then

$$D_{\vec{u}} f(x, y) = f_x(x, y) \cdot u_1 + f_y(x, y) \cdot u_2$$

Example

Let $f(x,y) = x^2 - xy + 3y^2$. Find the directional derivative of f in the direction $\langle 3, -4\rangle$ (at an arbitrary point (x,y)).

Step 1: Find a unit vector $\vec{u}$ in the direction $\langle 3,-4\rangle$

$$\vec{u} = \frac{1}{\|\langle 3,-4\rangle\|}\langle 3,-4\rangle = \frac{1}{\sqrt{9+16}}\langle 3,-4\rangle = \left\langle \frac{3}{5}, \frac{-4}{5} \right\rangle$$

Step 2: Compute the partial derivatives

$$f_x(x,y) = 2x - y \qquad f_y(x,y) = -x + 6y$$

Step 3: Combine: $D_{\vec{u}} f(x,y) = (2x-y)\cdot\frac{3}{5} + (-x+6y)\cdot\left(\frac{-4}{5}\right)$

Gradient

If $f(x,y)$ is differentiable, $\vec{u} = \langle u_1, u_2 \rangle$, $\|\vec{u}\| = 1$, then

$$D_{\vec{u}} f(x,y) = f_x(x,y) \cdot u_1 + f_y(x,y) \cdot u_2 \qquad (*)$$

$$= \langle f_x(x,y), f_y(x,y) \rangle \cdot \vec{u}$$

<u>Def</u>. Let $f(x,y)$ be a function of two variables such that f_x and f_y exist. Then the vector

$$\nabla f(x,y) := \langle f_x(x,y), f_y(x,y) \rangle$$

is called the gradient of f.

We can rewrite $(*)$ as

$$D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}$$

Examples

1. $f(x,y) = x^2 - xy + 3y^2$. Find $\nabla f(x,y)$.

$$f_x = 2x - y \qquad f_y = -x + 6y$$

$$\nabla f(x,y) = \langle 2x - y, -x + 6y \rangle = (2x - y)\cdot\vec{i} + (-x + 6y)\cdot\vec{j}$$

2. $f(x,y) = \sin(3x)\cos(3y)$. Find $\nabla f(x,y)$

$$f_x = 3\cos(3x)\cos(3y) \qquad f_y = -3\sin(3x)\sin(3y)$$

$$\nabla f(x,y) = \langle 3\cos(3x)\cos(3y), -3\sin(3x)\sin(3y) \rangle$$

Gradient as the direction of the steepest ascent

Consider a function $f(x,y)$ and a point (x_0,y_0).
We know that $D_{\vec{u}} f(x_0,y_0)$ gives the rate of change of function f at point (x_0,y_0) in the direction $\vec{u}$.

Q: For which $\vec{u}$ is $D_{\vec{u}} f(x_0,y_0)$ the greatest?

In other words, which direction gives the greatest rate of change?

Suppose that f is differentiable. Denote by φ the angle between $\nabla f(x_0,y_0)$ and $\vec{u}$

$$D_{\vec{u}} f(x_0,y_0) = \nabla f(x_0,y_0) \cdot \vec{u} = \|\nabla f(x_0,y_0)\| \overbrace{\|\vec{u}\|}^{=1} \cdot \cos\varphi$$

$$= \|\nabla f(x_0,y_0)\| \cdot \cos\varphi$$

Gradient as the direction of the steepest ascent

$$D_{\vec{u}} f(x_0, y_0) = \|\nabla f(x_0, y_0)\| \cdot \cos\varphi$$

Recall that $-1 \le \cos\varphi \le 1$, so

- $D_{\vec{u}} f(x_0, y_0)$ is maximized when $\cos\varphi = 1$, i.e., when $\vec{u}$ is in the same direction as $\nabla f(x_0, y_0)$

  In this case $\vec{u}_{max} = \dfrac{\nabla f(x_0, y_0)}{\|\nabla f(x_0, y_0)\|}$, and

  $$D_{\vec{u}_{max}} f(x_0, y_0) = \|\nabla f(x_0, y_0)\|$$

- $D_{\vec{u}} f(x_0, y_0)$ is minimized when $\cos\varphi = -1$, i.e., $\vec{u}_{min} = -\dfrac{\nabla f(x_0, y_0)}{\|\nabla f(x_0, y_0)\|}$

  $$D_{\vec{u}} f(x_0, y_0) = -\|\nabla f(x_0, y_0)\|$$

- If $\nabla f(x_0, y_0) = (0, 0)$, then $D_{\vec{u}} f(x_0, y_0) = 0$ for any direction $\vec{u}$

Example

Find the direction for which the directional derivative of $f(x,y) = 2x^2 - xy + 3y^2$ at $(-2,3)$ is a maximum. What is the maximum value?

First, compute the gradient at $(x_0, y_0) = (-2, 3)$

$$f_x = 4x - y \qquad f_y = -x + 6y$$

$$f_x(-2,3) = 4\cdot(-2) - 3 = -11, \quad f_y(-2,3) = -(-2) + 6\cdot 3 = 20$$

$$\nabla f(-2,3) = \langle -11, 20 \rangle$$

$$\|\nabla f(-2,3)\| = \sqrt{11^2 + 20^2} = \sqrt{521}$$

The direction of the most rapid increase: $\left\langle \frac{-11}{\sqrt{521}}, \frac{20}{\sqrt{521}} \right\rangle$

The rate of change in this direction is $\sqrt{521}$.