# MATH 10C: Calculus III (Lecture B00)

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## Today: Differentiability. Chain rule

## Next: Strang 4.5

Week 6:

- homework 5 (due Friday, November 4, 11:59 PM)

# Differentiability

Functions of one variable: if a function is differentiable at $x_0$, the graph at $x_0$ is smooth (no corners), tangent line is well defined and approximates well the function around $x_0$.

Functions of two variables: differentiability gives the condition when the surface at $(x_0, y_0)$ is smooth, by which we mean that the tangent plane at $(x_0, y_0)$ exists.

Notice, that whenever $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist, we can always write the equation

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \qquad (*)$$

But this does not mean that the tangent plane exists (if it exists, it is given by $(*)$).

# Differentiability

**Def.** $f$ is differentiable at $(x_0, y_0)$ if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and the error term

$$E(x,y) = f(x,y) - [\, f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \,]$$

satisfies

This means that

$$f(x,y) =$$

and $E(x,y)$ goes to zero faster than the distance between $(x,y)$ and $(x_0, y_0)$.

**Remark** If $f(x,y)$ is differentiable at $(x_0,y_0)$, then $f(x,y)$ is continuous at $(x_0,y_0)$.

# Differentiability

The existence of partial derivatives is not sufficient to have differentiability.

Example

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = 0 \end{cases}$$

Then $f_x(x,y) = \Big\{$ , $f_y(x,y) = \Big\{$

For $(x_0,y_0) = (0,0)$, $f(0,0) =$ , $f_x(0,0) =$ , $f_y(0,0) =$ , so

$E(x,y) =$ . , and

.

## Differentiability

But, if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist AND are continuous in a neigborhood of $(x_0, y_0)$, then $f$ is differentiable at $(x_0, y_0)$

### Theorem

If $f(x, y)$, $f_x(x, y)$, $f_y(x, y)$ all exist in a neighborhood of $(x_0, y_0)$ and

## The chain rule

Recall that for functions of one variable

$$\frac{d}{dx}(f(g(x))) = f'(g(x))\,g'(x)$$

**Thm** (Chain rule for one independent variable)

Let $x(t)$ and $y(t)$ be differentiable functions, let $f: \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function. Then

$$\frac{d}{dt}[f(x(t), y(t))] =$$

**Example** Compute $\frac{d}{dt}[f(\sin t, \cos t)]$ with $f(x,y) = 4x^2 + 3y^2$

$$\frac{\partial f}{\partial x} = \quad , \quad \frac{\partial f}{\partial y} = \quad , \quad \frac{d}{dt}\sin t = \quad , \quad \frac{d}{dt}\cos t =$$

$$\frac{d}{dt}[f(\sin t, \cos t)] =$$

# The chain rule

**Thm** (Chain rule for two independent variables)

Suppose $x(u,v)$ and $y(u,v)$ are differentiable, and suppose $f(x,y)$ is differentiable. Then

$z = f(x(u,v), y(u,v))$ is differentiable (function from $\mathbb{R}^2$ to $\mathbb{R}$)

and

$$\frac{\partial z}{\partial u} =$$

$$\frac{\partial z}{\partial v} =$$

**Example** $z = f(x,y) = e^{x^2+3y}$, $x(u,v) = u+2v$, $y(u,v) = u-v$

$\frac{\partial f}{\partial x} =$ , $\frac{\partial f}{\partial y} =$ , $\frac{\partial x}{\partial u} =$ , $\frac{\partial x}{\partial v} =$ , $\frac{\partial y}{\partial u} =$ , $\frac{\partial y}{\partial v} =$

$$\frac{\partial z}{\partial u} =$$