MATH 10C: Calculus III (Lecture B00)

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Today: Partial derivatives

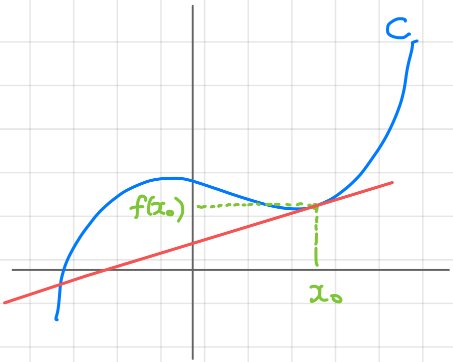
Next: Strang 4.4

Week 6:

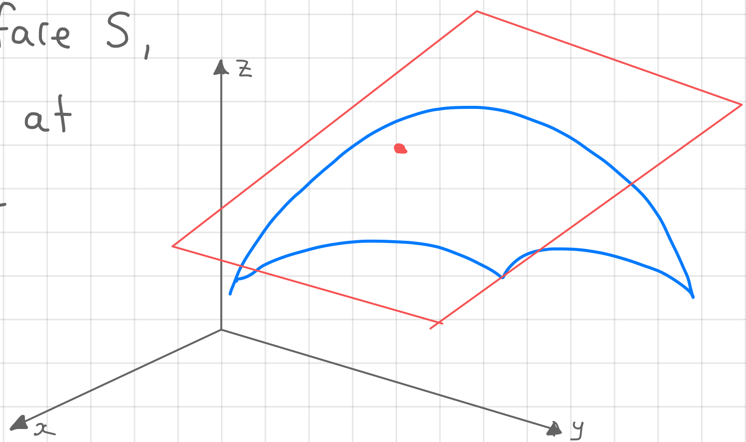
- homework 5 (due Friday, November 4, 11:59 PM)

Tangent planes

Recall, if f is a function of one real variable, then its graph determines a curve C in $\mathbb{R}^2$, and the tangent line to the graph of f at point x_0 is the line that "touches" the curve C at point $(x_0, f(x_0))$

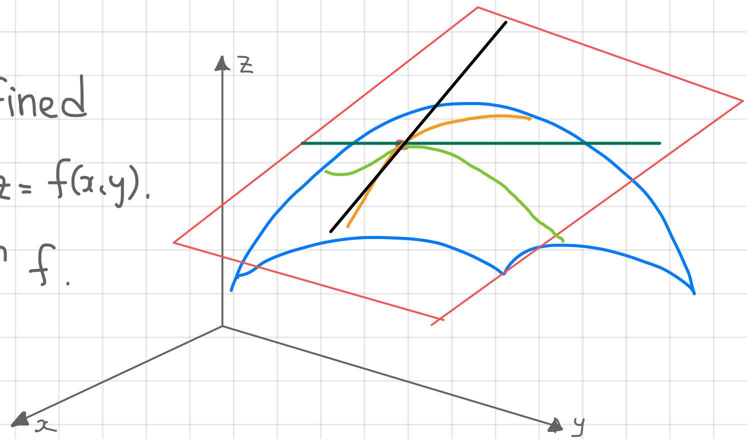


If f is a function of two variables, then its graph determines a surface S, and the tangent plane to S at $(x_0, y_0, f(x_0, y_0))$ is a plane that "touches" S at this point.



Tangent plane

Def. Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface S, and let C be any curve passing through P_0 and lying entirely in S. If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the



Def. Let S be a surface defined by a differentiable function $z = f(x,y)$. Let $P_0 = (x_0, y_0)$ be in the domain of f. Then the equation of the tangent plane to S at P_0 is

Tangent plane

To see that this formula is correct, we can find two curves in S that pass through $(x_0, y_0, f(x_0,y_0))$ and determine the equations of the tangent lines.

Take $\vec{p}(t) =$ and $\vec{q}(s) =$

Then for any t (such that (t, y_0) is in the domain of f)

$\vec{p}(t)$. Similarly, for any s $\vec{q}(s)$

. Moreover,

Tangent line to $\vec{p}(t)$ at $t = x_0$: $\vec{\ell}_p(t) =$

with $\vec{p}\,'(t) =$

Similarly, tangent line to $\vec{q}(s)$ at $s = y_0$: $\vec{\ell}_q(s) =$

$\vec{q}\,'(s) =$

Tangent plane

Vectors $\vec{p}\,'(x_0) = \langle 1, 0, \frac{\partial f}{\partial x}(x_0, y_0) \rangle$ and $\vec{q}\,'(y_0) = \langle 0, 1, \frac{\partial f}{\partial y}(x_0, y_0) \rangle$ are not parallel, therefore, together with the point $(x_0, y_0, f(x_0, y_0))$ they determine a plane with normal vector

$\vec{n} =$

The equation of a plane passing through $(x_0, y_0, f(x_0, y_0))$ with normal vector $\vec{n}$ is

Tangent plane

Example Find the equation of the tangent plane to the surface defined by the function $f(x,y) = e^{xy}$ at point $(1,-1)$

- Step 1: Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial x} = \qquad\qquad \frac{\partial f}{\partial y} =$$

- Step 2: Evaluate $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$

$$\frac{\partial f}{\partial x}(1,-1) = \qquad\qquad \frac{\partial f}{\partial y}(1,-1) =$$

- Step 3: Evaluate $f(x_0, y_0)$: $f(1,-1) =$

- Step 4: Plug everything into the equation:

Tangent plane does not always exist at every point

Example (tangent plane does not exist at $(0,0)$)

Let $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} , & (x,y) \neq 0 \\ 0 , & (x,y) = 0 \end{cases}$

($f(x,y)$ is continuous)

S - surface defined by $f(x,y)$

Consider the curves:

Consider the curve $\vec{p}(t) =$

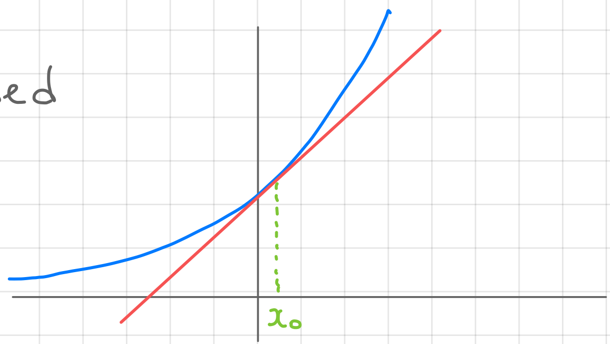
Then $f(t,t) =$

For a tangent plane to a surface to exist, it is sufficient that the function that defines the surface is differentiable.

Linear approximation

Functions of one variable:

$$y = f(x_0) + f'(x_0)(x - x_0)$$



the tangent line at x_0 can be used as the linear approximation of a function $f(x)$ at points x close to x_0:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \quad \text{for } x \text{ close to } x_0$$

<u>Functions of two variables:</u> the tangent plane at (x_0, y_0) can be used as the linear approximation of $f(x,y)$ at points close to (x_0, y_0)

<u>Def.</u> Given a function $z = f(x,y)$ with continuous partial derivatives that exist at (x_0, y_0), the linear approximation of f at point (x_0, y_0) is given by

Linear approximation

Example

Given function $f(x,y) = e^{2x-y-1}$ approximate $f(1.01, 0.99)$ using points $(1,1)$ as (x_0, y_0).

- Compute the derivatives

  $f_x(x,y) =$, $f_y(x,y) =$

- Evaluate f, f_x and f_y at (x_0, y_0)

  $f(1,1) =$, $f_x(1,1) =$, $f_y(1,1) =$

- Write the linear approximation

  $L(x,y) =$

- Compute the approximation: $L(1.01, 0.99) =$

Differentiability

Functions of one variable: if a function is differentiable at x_0, the graph at x_0 is smooth (no corners), tangent line is well defined and approximates well the function around x_0.

Functions of two variables: differentiability gives the condition when the surface at (x_0, y_0) is smooth, by which we mean that the tangent plane at (x_0, y_0) exists.

Notice, that whenever $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist, we can always write the equation

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \qquad (*)$$

But this does not mean that the tangent plane exists (if it exists, it is given by $(*)$).

Differentiability

Def. f is differentiable at (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and the error term

$$E(x,y) = f(x,y) - [\, f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \,]$$

satisfies

This means that

$$f(x,y) =$$

and $E(x,y)$ goes to zero faster than the distance between (x,y) and (x_0, y_0).

Remark If $f(x,y)$ is differentiable at (x_0,y_0), then $f(x,y)$ is continuous at (x_0,y_0).

Differentiability

The existence of partial derivatives is not sufficient to have differentiability.

Example

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = 0 \end{cases}$$

Then $f_x(x,y) = \Bigg\{$, $f_y(x,y) = \Bigg\{$

For $(x_0,y_0) = (0,0)$, $f(0,0) =$, $f_x(0,0) =$, $f_y(0,0) =$, so

$E(x,y) =$, and

Differentiability

But, if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist AND are continuous in a neigborhood of (x_0, y_0), then f is differentiable at (x_0, y_0)

Theorem

If $f(x,y)$, $f_x(x,y)$, $f_y(x,y)$ all exist in a neighborhood of (x_0, y_0) and

The chain rule

Recall that for functions of one variable

$$\frac{d}{dx}(f(g(x))) = f'(g(x))\,g'(x)$$

Thm (Chain rule for one independent variable)

Let $x(t)$ and $y(t)$ be differentiable functions, let $f: \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function. Then

$$\frac{d}{dt}[f(x(t), y(t))] =$$

Example Compute $\frac{d}{dt}[f(\sin t, \cos t)]$ with $f(x,y) = 4x^2 + 3y^2$

$$\frac{\partial f}{\partial x} = \quad , \quad \frac{\partial f}{\partial y} = \quad , \quad \frac{d}{dt}\sin t = \quad , \quad \frac{d}{dt}\cos t =$$

$$\frac{d}{dt}[f(\sin t, \cos t)] =$$

The chain rule

Thm (Chain rule for two independent variables)

Suppose $x(u,v)$ and $y(u,v)$ are differentiable, and suppose $f(x,y)$ is differentiable. Then

$z = f(x(u,v), y(u,v))$ is differentiable (function from $\mathbb{R}^2$ to $\mathbb{R}$)

and

$$\frac{\partial z}{\partial u} =$$

$$\frac{\partial z}{\partial v} =$$

Example $z = f(x,y) = e^{x^2+3y}$, $x(u,v) = u+2v$, $y(u,v) = u-v$

$\frac{\partial f}{\partial x} =$, $\frac{\partial f}{\partial y} =$, $\frac{\partial x}{\partial u} =$, $\frac{\partial x}{\partial v} =$, $\frac{\partial y}{\partial u} =$, $\frac{\partial y}{\partial v} =$

$$\frac{\partial z}{\partial u} =$$