MATH 10C: Calculus III (Lecture B00)

mathweb.ucsd.edu/~ynemish/teaching/10c

Today: Partial derivatives

Next: Strang 4.4

Week 6:

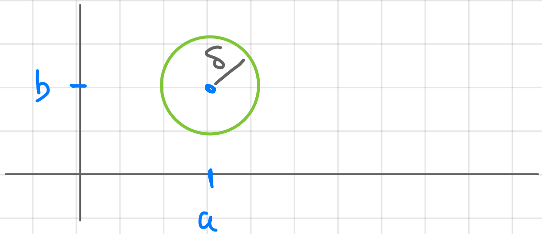
- homework 5 (due Friday, November 4, 11:59 PM)

Limit of a function of two variables

ε, δ error tolerance

__Def__ Consider a point $(a,b) \in \mathbb{R}^2$. A δ-disk centered at point (a,b) is the open disk of radius δ centered at (a,b)

$$\{ (x,y) \mid (x-a)^2 + (y-b)^2 < \delta^2 \}$$



__Def__. The limit of $f(x,y)$ as (x,y) approaches (x_0, y_0) is L

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if for each $\varepsilon > 0$ there exists a small enough $\delta > 0$ such that all points in a δ-disk around (x_0,y_0), except possible (x_0,y_0) itself, $f(x,y)$ is no more than ε away from L. (For any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x,y) - L| < \varepsilon$ whenever $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$.)

Computing limits. Limit laws

Theorem 4.1 Let $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, $\lim\limits_{(x,y)\to(a,b)} g(x,y) = M$, c - constant

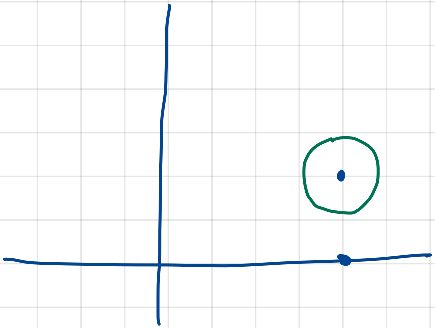
- $\lim\limits_{(x,y)\to(a,b)} c = c$
- $\lim\limits_{(x,y)\to(a,b)} x = a$
- $\lim\limits_{(x,y)\to(a,b)} y = b$
- $\lim\limits_{(x,y)\to(a,b)} [f(x,y) \pm g(x,y)] = L \pm M$
- $\lim\limits_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = LM$
- If $M \neq 0$, $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$
- $\lim\limits_{(x,y)\to(a,b)} [c\, f(x,y)] = cL$
- $\lim\limits_{(x,y)\to(a,b)} [f(x,y)]^n = L^n$
- $\lim\limits_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$

Examples

$$\lim_{(x,y)\to(0,0)} \frac{xy+1}{x^2+y^2+1} = \frac{0\cdot 0+1}{0^2+0^2+1} = 1$$

$\dfrac{1}{x^2+y^2+1}$ is continuous everywhere

$$\lim_{(x,y)\to(2,1)} \frac{x-y-1}{\sqrt{x-y}-1} = \lim_{(x,y)\to(2,1)} \frac{\left(\sqrt{x-y}\right)^2-1}{\sqrt{x-y}-1} = \lim_{(x,y)\to(2,1)} \frac{\cancel{\left(\sqrt{x-y}-1\right)}\left(\sqrt{x-y}+1\right)}{\cancel{\sqrt{x-y}-1}}$$



$$a^2-b^2=(a-b)(a+b)$$

$$= \lim_{(x,y)\to(2,1)} \left(\sqrt{x-y}+1\right) = \sqrt{2-1}+1 = 2$$

Partial derivatives of functions of two variables

Functions of one variable $y = f(x)$: the derivative gives the instantaneous rate of change of y as a funtion of x.

Functions of two variables $z = f(x,y)$ have 2 independent variables, we need two (partial) derivatives.

Def The partial derivative of $f(x,y)$ with respect to x is

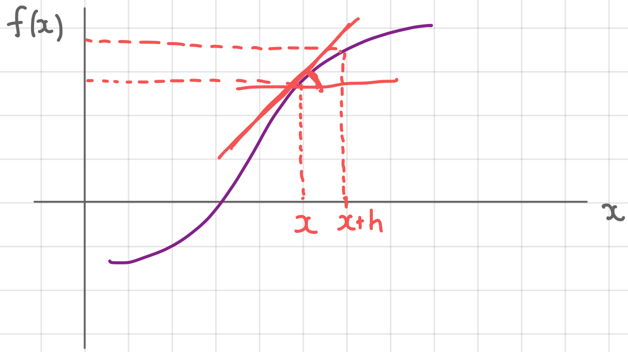
$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

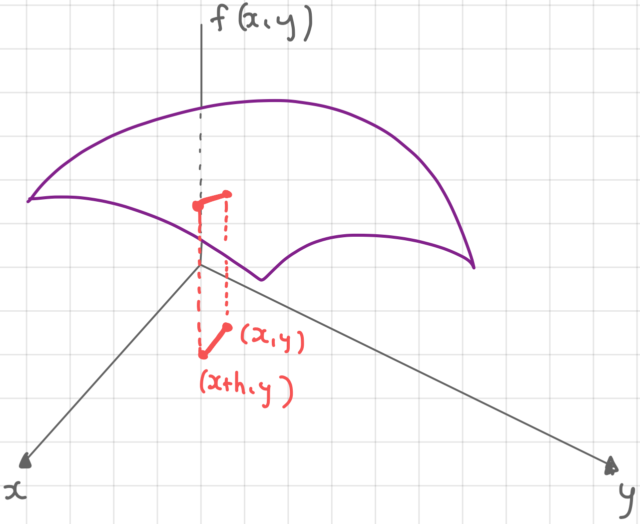
The partial derivative of $f(x,y)$ with respect to y is

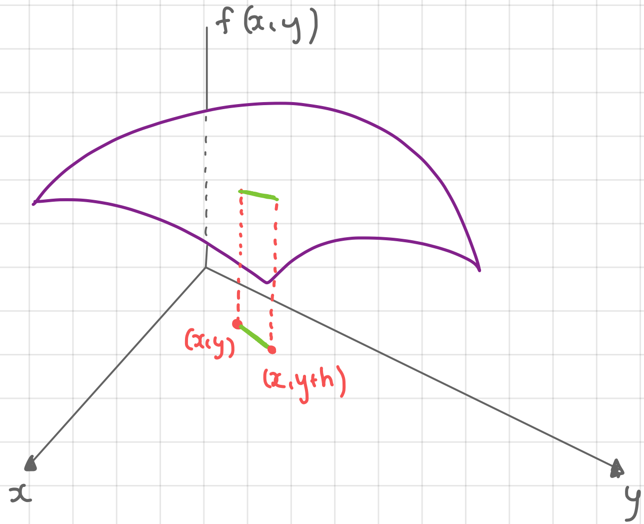
$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

Partial derivatives of functions of two variables

Partial derivatives measure the instantaneous rate of change of f if we change only the x variable $\frac{\partial f}{\partial x}$ or only the y variable $\frac{\partial f}{\partial y}$







Calculating partial derivatives

Rule To differentiate $f(x,y)$ with respect to x, treat the variable y as a constant, and differentiate f as a function of one variable x:

$$\frac{\partial}{\partial x}\left(x^3 - 12xy^2 - x^2y + 4x - y - 3\right) = 3x^2 - 12y^2 - 2xy + 4$$

To differentiate $f(x,y)$ with respect to y, treat the variable x as a constant, and differentiate f as a function of one variable y:

$$\frac{\partial}{\partial y}\left(x^3 - 12xy^2 - x^2y + 4x - y - 3\right) = -24xy - x^2 - 1$$

$$\frac{\partial}{\partial y}(x^2y) = \lim_{h\to 0}\frac{x^2(y+h) - x^2y}{h} = \lim_{h\to 0}\frac{\cancel{x^2y} + x^2h - \cancel{x^2y}}{h} = x^2\lim_{h\to 0}\frac{h}{h} = x^2$$

Calculating partial derivatives

Example $f(x,y) = e^{-\frac{x^2+y^2}{2}}$

$$\left(-\frac{x^2+a}{2}\right)' = -x$$

Compute $\dfrac{\partial f}{\partial x} = e^{-\frac{x^2+y^2}{2}}(-x) = -x\, e^{-\frac{x^2+y^2}{2}}$

$$\frac{\partial f}{\partial y} = e^{-\frac{x^2+y^2}{2}}(-y) = -y\, e^{-\frac{x^2+y^2}{2}}$$

Higher-order partial derivatives

Each partial derivative is itself a function of two variables, so we can compute their partial derivatives, which we call higher-order partial derivatives. For example, there are 4 second-order partial derivatives

$$\underset{\displaystyle f_{xx}}{\underset{\parallel}{\frac{\partial^2 f}{\partial x^2}}} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right], \quad \underset{\displaystyle f_{yx}}{\underset{\parallel}{\frac{\partial^2 f}{\partial y\,\partial x}}} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right], \quad \underset{\displaystyle f_{xy}}{\underset{\parallel}{\frac{\partial^2 f}{\partial x\,\partial y}}} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right], \quad \underset{\displaystyle f_{yy}}{\underset{\parallel}{\frac{\partial^2 f}{\partial y^2}}} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]$$

f_{xy} and f_{yx} are called **mixed partial derivatives**

f_{xy} and f_{yx} are not necessarily equal.

<u>Thm</u> If f_{xy} and f_{yx} are continuous on an open disk D, then $f_{xy} = f_{yx}$ on D.

Higher-order partial derivatives

Example Let $f(x,y) = x e^{-y^2}$

$$\frac{\partial f}{\partial x} = e^{-y^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = e^{-y^2} \cdot (-2y)$$

$$\frac{\partial f}{\partial y} = x e^{-y^2} \cdot (-2y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^{-y^2} \cdot (-2y)$$

It is not true in general that $f_{xy} = f_{yx}$.