MATH 10C: Calculus III (Lecture B00)

mathweb.ucsd.edu/~ynemish/teaching/10c

Today: Partial derivatives

Next: Strang 4.4

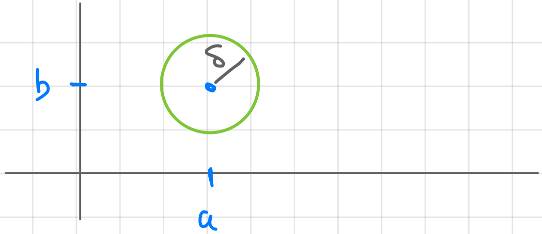
Week 5:

- homework 4 (due Friday, October 28)
- regrades of Midterm 1 on Gradescope until October 30

Limit of a function of two variables

Def Consider a point $(a,b) \in \mathbb{R}^2$. A δ-disk centered at point (a,b) is the open disk of radius δ centered at (a,b)

$$\{ (x,y) \mid (x-a)^2 + (y-b)^2 < \delta^2 \}$$



Def. The limit of $f(x,y)$ as (x,y) approaches (x_0, y_0) is L

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

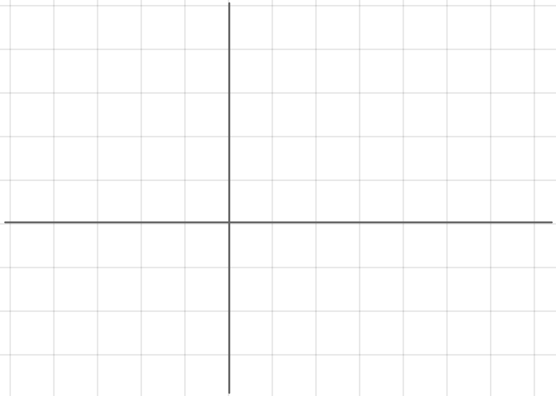
if for each $\varepsilon > 0$ there exists a small enough $\delta > 0$ such that all points in a δ-disk around (x_0,y_0), except possible (x_0,y_0) itself, $f(x,y)$ is no more than ε away from L. (For any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x,y) - L| < \varepsilon$ whenever $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$.)

Limit of a function of two variables

This definition ensures that if $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$, then

any way of approaching (x_0, y_0) results in the same limit L.

(Another) example when the limit fails to exist:



- approach $(0,0)$ along the

- approach $(0,0)$ along the curve

Computing limits. Limit laws

Theorem 4.1 Let $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, $\lim\limits_{(x,y)\to(a,b)} g(x,y) = M$, c - constant

- $\lim\limits_{(x,y)\to(a,b)} c =$
- $\lim\limits_{(x,y)\to(a,b)} x =$
- $\lim\limits_{(x,y)\to(a,b)} y =$
- $\lim\limits_{(x,y)\to(a,b)} [f(x,y) \pm g(x,y)] =$
- $\lim\limits_{(x,y)\to(a,b)} [f(x,y)g(x,y)] =$
- , $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} =$
- $\lim\limits_{(x,y)\to(a,b)} [c\, f(x,y)] =$
- $\lim\limits_{(x,y)\to(a,b)} [f(x,y)]^n =$
- $\lim\limits_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} =$

Computing limits. Limit laws

Examples

$$\lim_{(x,y)\to(1,2)} \frac{\sqrt{3x-y}}{(x^2+xy+y^3)^2}$$

$$= \frac{\lim\limits_{(x,y)\to(1,2)} \sqrt{3x-y}}{\lim\limits_{(x,y)\to(1,2)} (x^2+xy+y^3)^2}$$

$$= \frac{\sqrt{\lim\limits_{(x,y)\to(1,2)} [3x-y]}}{\left(\lim\limits_{(x,y)\to(1,2)} [x^2+xy+y^3]\right)^2}$$

$$= \frac{\sqrt{3\lim\limits_{(x,y)\to(1,2)} x - \lim\limits_{(x,y)\to(1,2)} y}}{\left(\left(\lim\limits_{(x,y)\to(1,2)} x\right)^2 + \left(\lim\limits_{(x,y)\to(1,2)} x\right)\left(\lim\limits_{(x,y)\to(1,2)} y\right) + \left(\lim\limits_{(x,y)\to(1,2)} y\right)^3\right)^2}$$

$$= \frac{\sqrt{3\cdot 1-2}}{(1^2+1\cdot 2+2^3)^2} = \frac{1}{11^2} = \frac{1}{121}$$

Continuity of functions of two variables

Def. A function $f(x,y)$ is continuous at a point (a,b) if

(i)

(ii)

(iii)

Properties

1. If $f(x,y)$ and $g(x,y)$ are continuous at (x_0,y_0), then

is continuous at (x_0,y_0)

2. If $\varphi(x)$ is continuous at x_0 and $\psi(y)$ is continuous at y_0, then

is continuous at (x_0,y_0)

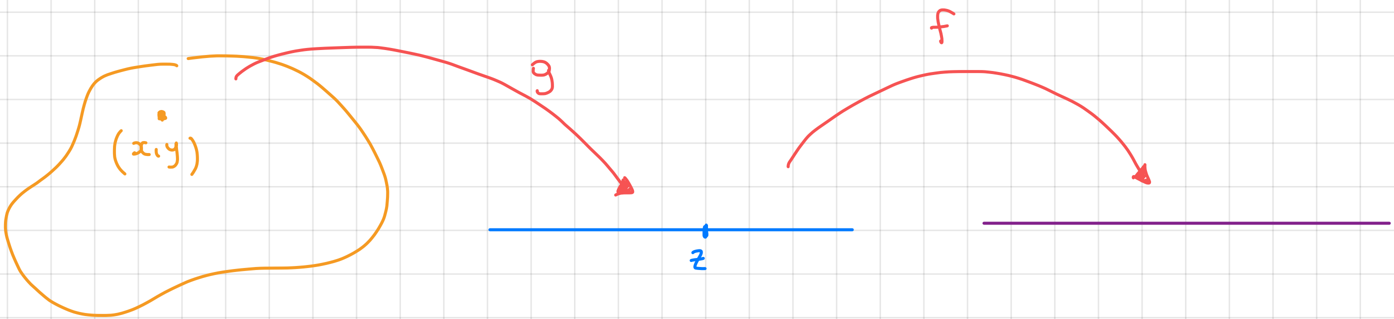
Continuity of functions of two variables

Properties (cont.)

3. If $g(x,y)$ is continuous at (x_0, y_0), and $f(z)$ is continuous at $z_0 := g(x_0, y_0)$, then

$$f \circ g$$

is continuous at (x_0, y_0)



Continuity of functions of two variables

Example $\dfrac{\sqrt{3x-y}}{(x^2+xy+y^3)^2}$: $3x-y$ is continuous on

$\sqrt{z}$ is continuous for

so $\sqrt{3x-y}$ is continuous for

Similarly, x^2+xy+y^3 is continuous on

$\dfrac{1}{z^2}$ is continuous for all

so $\dfrac{1}{(x^2+xy+y^3)^2}$ is continuous at

Take $(x_0,y_0)=(1,2)$. Then , so both

f_1 and f_2 are

Partial derivatives of functions of two variables

Functions of one variable $y = f(x)$: the derivative gives the instantaneous rate of change of y as a funtion of x.

Functions of two variables $z = f(x,y)$ have 2 independent variables, we need two (partial) derivatives.

Def The partial derivative of $f(x,y)$ with respect to x

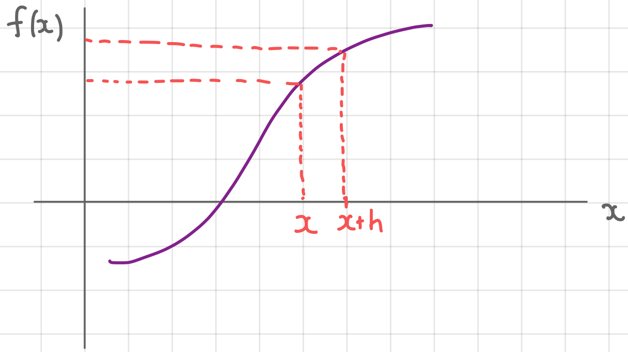
is $$f_x = \frac{\partial f}{\partial x} =$$

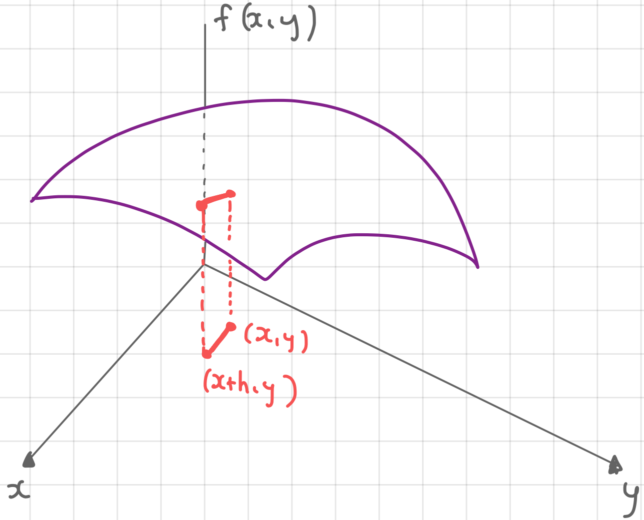
The partial derivative of $f(x,y)$ with respect to y

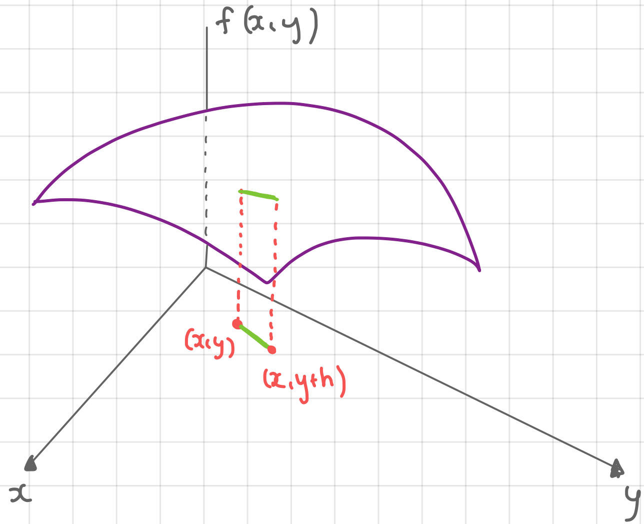
is $$f_y = \frac{\partial f}{\partial y} =$$

Partial derivatives of functions of two variables

Partial derivatives measure the instantaneous rate of change of f if we change only the x variable $\frac{\partial f}{\partial x}$ or only the y variable $\frac{\partial f}{\partial y}$







Calculating partial derivatives

Rule To differentiate $f(x,y)$ with respect to x, treat the variable y as a constant, and differentiate f as a function of one variable x:

$$\frac{\partial}{\partial x}\left(x^3 - 12xy^2 - x^2y + 4x - y - 3\right) =$$

To differentiate $f(x,y)$ with respect to y, treat the variable x as a constant, and differentiate f as a function of one variable y:

$$\frac{\partial}{\partial y}\left(x^3 - 12xy^2 - x^2y + 4x - y - 3\right) =$$

Calculating partial derivatives

Example $f(x,y) = e^{-\frac{x^2+y^2}{2}}$

Compute $\dfrac{\partial f}{\partial x} =$

$\dfrac{\partial f}{\partial y} =$

Higher-order partial derivatives

Each partial derivative is itself a function of two variables, so we can compute their partial derivatives, which we call higher-order partial derivatives. For example, there are 4 second-order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = \qquad , \quad \frac{\partial^2 f}{\partial y \partial x} = \qquad , \quad \frac{\partial^2 f}{\partial x \partial y} = \qquad , \quad \frac{\partial^2 f}{\partial y^2} =$$

f_{xy} and f_{yx} are called

f_{xy} and f_{yx} are not necessarily equal.

<u>Thm</u> If f_{xy} and f_{yx} are continuous on an open disk D, then $f_{xy} = f_{yx}$ on D.

Higher-order partial derivatives

Example Let $f(x,y) = x^2\cos(2x-y) + xe^{-y^2}$

$$\frac{\partial f}{\partial x} =$$

$$\frac{\partial^2 f}{\partial y \partial x} =$$

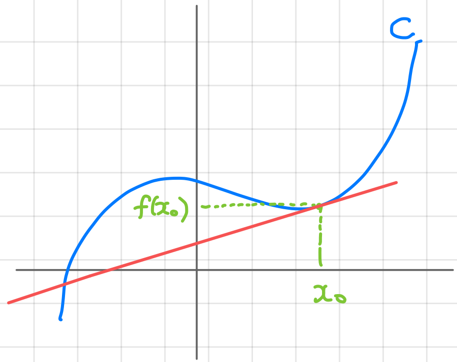
$$\frac{\partial f}{\partial y} =$$

$$\frac{\partial^2 f}{\partial x \partial y} =$$

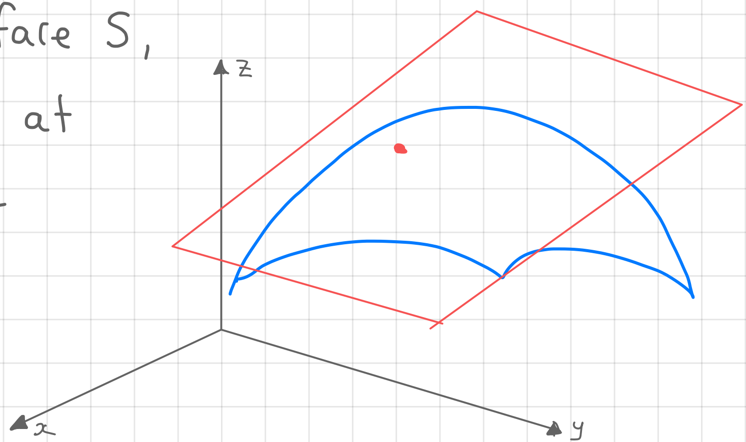
It is not true in general that $f_{xy} = f_{yx}$.

Tangent planes

Recall, if f is a function of one real variable, then its graph determines a curve C in $\mathbb{R}^2$, and the tangent line to the graph of f at point x_0 is the line that "touches" the curve C at point $(x_0, f(x_0))$

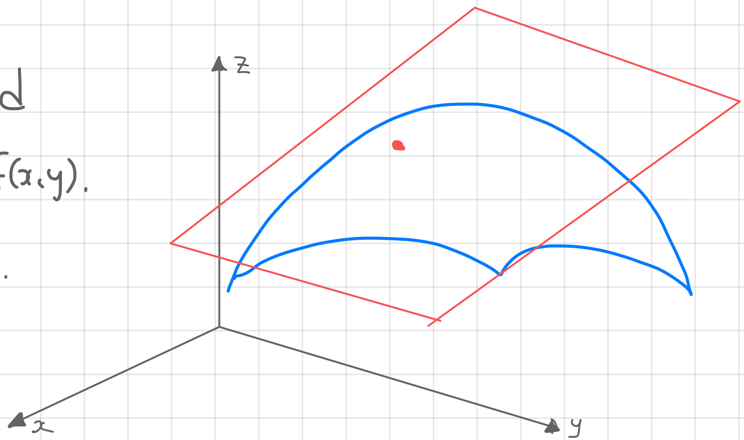


If f is a function of two variables, then its graph determines a surface S, and the tangent plane to S at $(x_0, y_0, f(x_0,y_0))$ is a plane that "touches" S at this point.



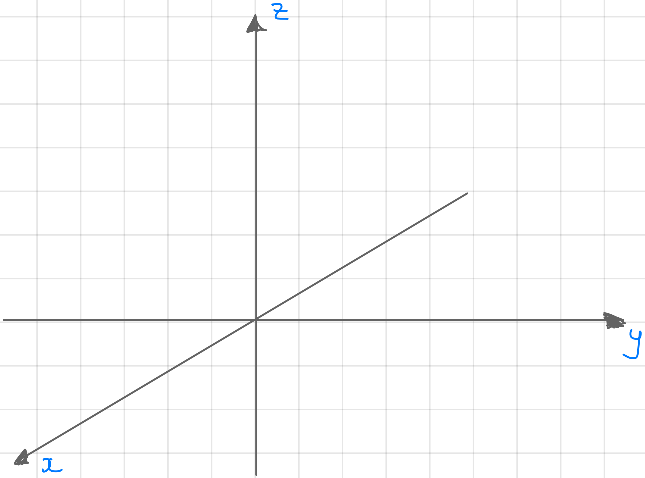
Tangent plane

Def. Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface S, and let C be any curve passing through P_0 and lying entirely in S. If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the

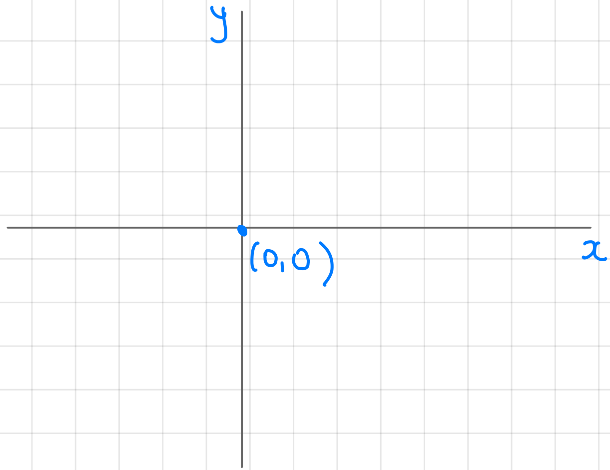


Def. Let S be a surface defined by a differentiable function $z = f(x,y)$.
Let $P_0 = (x_0, y_0)$ be in the domain of f.
Then the equation of the tangent plane to S at P_0 is

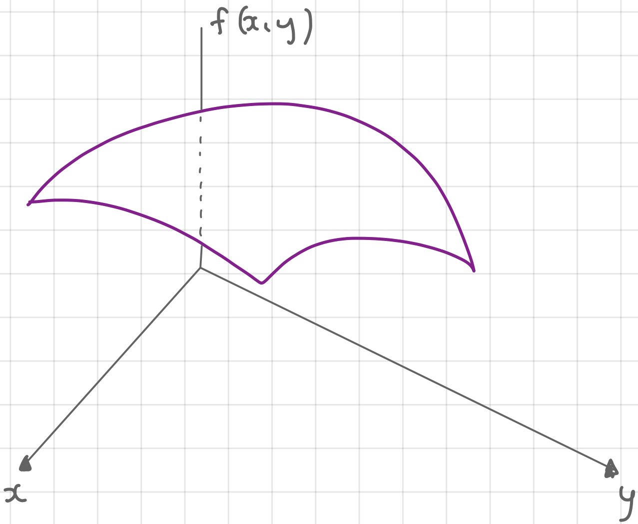
$\vec{v}_0$
$\vec{r}_0 = \langle 0\ 0 \rangle$

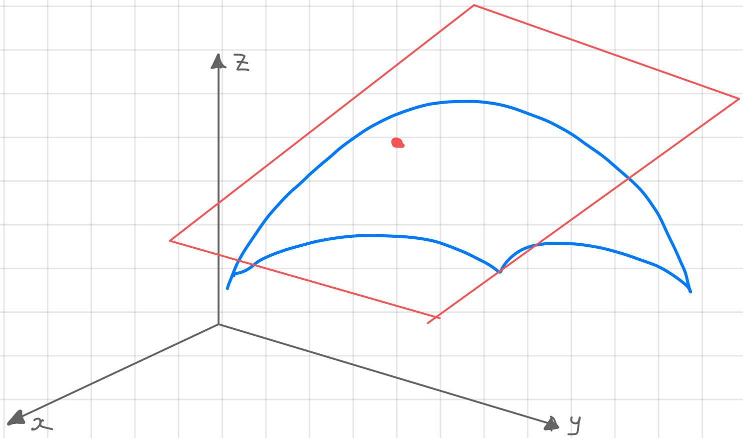


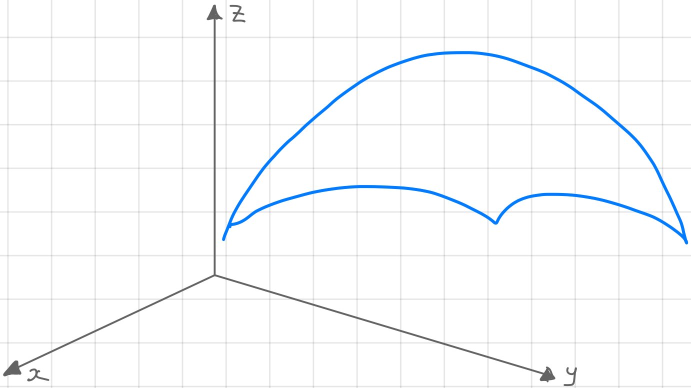
z
y
x



y
x
(0,0)


f (x, y)
x
y


z
x
y


z
x
y