MATH 10C: Calculus III (Lecture B00)

mathweb.ucsd.edu/~ynemish/teaching/10c

Today: Functions of two variables

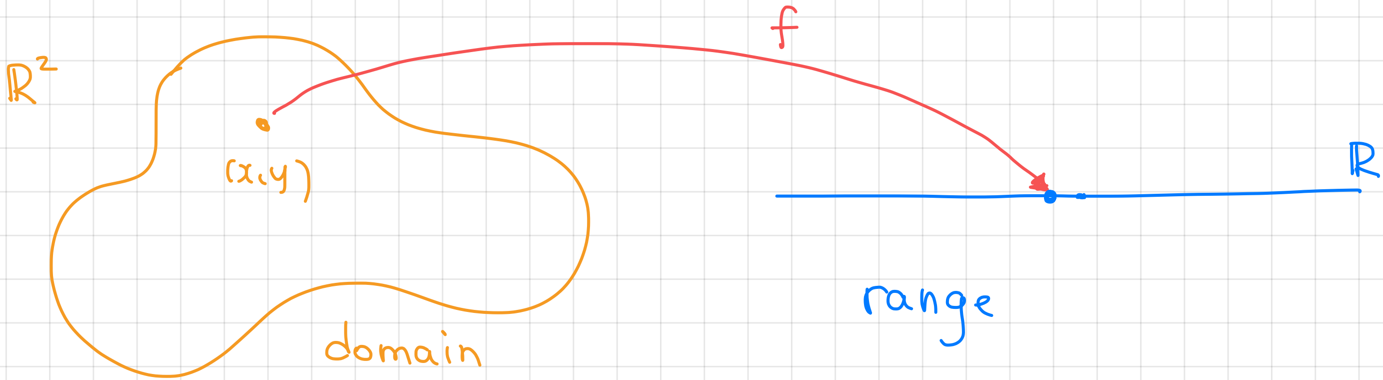
Next: Strang 4.2

Week 5:

- homework 4 (due Friday, October 27)
- regrades of Midterm 1 on Gradescope until October 29

Functions of two variables

Def. A function of two variables maps each ordered pair (x,y) in a subset $D \subset \mathbb{R}^2$ to a unique real number $z = f(x,y)$. The set D is called the domain of the function. The range of f is the set of all real numbers z that has at least one ordered pair $(x,y) \in D$ s.t. $f(x,y) = z$.



If not specified, we choose the domain to be the set of all pairs (x,y) for which $f(x,y)$ is well-defined.

Functions of two variables

Example Find the domain and range of the function

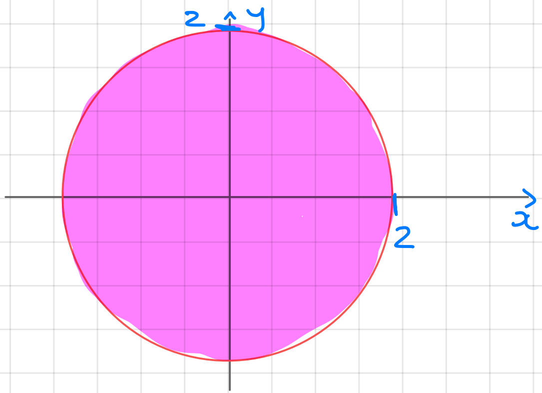
$$f(x,y) = \sqrt{4 - x^2 - y^2}$$

Domain. One restriction: the number under the square root has to be nonnegative, i.e.,

The set of all pairs $(x,y) \in \mathbb{R}^2$ such that

is a

The domain of f is



Range. For (x,y) in the domain

the range of x^2+y^2 is

the range of $4-x^2-y^2$ is

the range of $\sqrt{4-x^2-y^2}$ is

Graph of a function of two variables

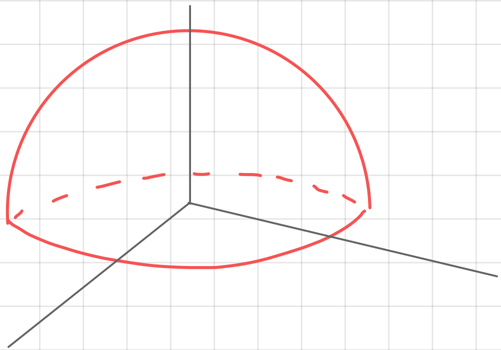
Function f of two variables: maps each pair (x,y) from its domain to a real number $z = f(x,y)$.

The graph of f consists of ordered triples $(x,y,f(x,y))$ for all (x,y) in the domain of f. We call the graph of a function of two variables a surface.

Example $f(x,y) = \sqrt{4-x^2-y^2}$, $\mathrm{dom}(f) = \{(x,y) \mid x^2+y^2 \leq 4\}$

Graph of f consists of all $(x,y,z) \in \mathbb{R}^3$ such that $z = \sqrt{4-x^2-y^2}$, or

– equation of a



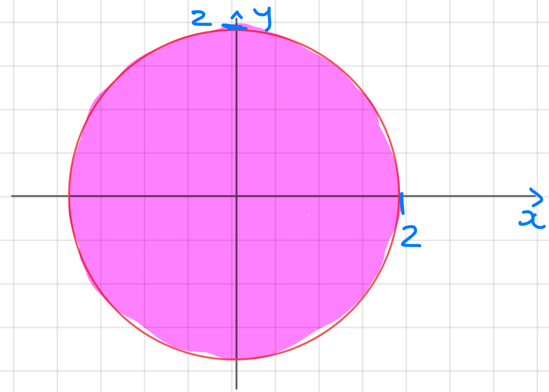
Level curves

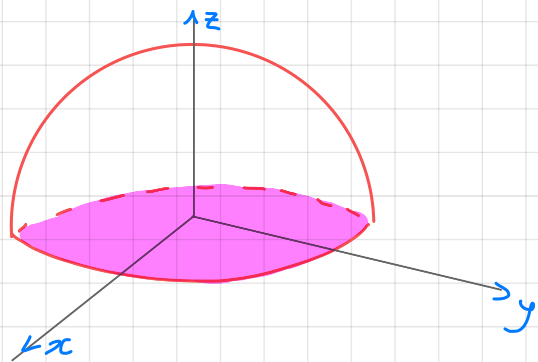
Def. Given a function $f(x,y)$ and a number c in the range of f, a level curve of a function of two variables for the value c is defined to be

Example $f(x,y) = \sqrt{4 - x^2 - y^2}$

Range of f is $[0, 2]$.

Take . Then the level curve of f for is defined by





Functions of more than two variables

In a similar way we can define functions of more than two variables, e.g., functions of three variables:

to each point (x, y, z) in the domain assign a real number $f(x, y, z)$.

<u>Example</u> $f(x,y,z) =$; domain: all points $(x,y,z) \in \mathbb{R}$

such that , i.e.

Limit of a function of two variables

Informally

means that $f(x,y)$ gets closer to L as (x,y) gets closer to (x_0, y_0).

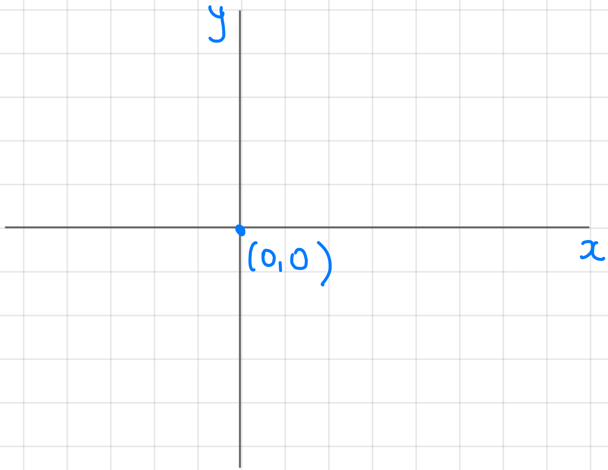
We have to be careful with what "(x,y) gets closer to (x_0, y_0)" means!

For example,

let

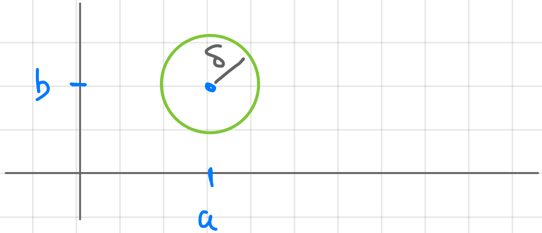
- If (x,y) approaches $(0,0)$ along the line

- If (x,y) approaches $(0,0)$ along the line



Limit of a function of two variables

Def Consider a point $(a,b) \in \mathbb{R}^2$. A δ-disk centered at point (a,b) is the open disk of radius δ centered at (a,b)



Def. The limit of $f(x,y)$ as (x,y) approaches (x_0, y_0) is L

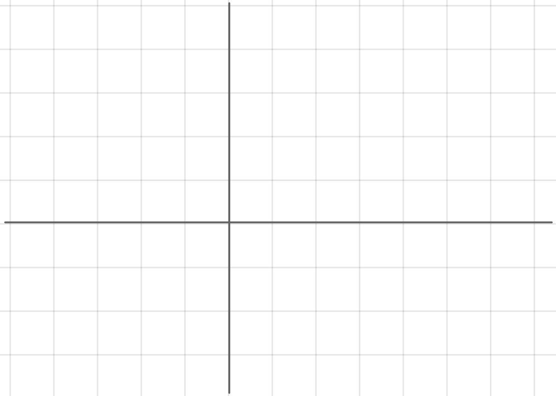
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if for each $\varepsilon > 0$ there exists a small enough $\delta > 0$ such that all points in a δ-disk around (x_0,y_0), except possible (x_0,y_0) itself, $f(x,y)$ is no more than ε away from L.

Limit of a function of two variables

This definition ensures that if $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$, then any way of approaching (x_0,y_0) results in the same limit L.

(Another) example when the limit fails to exist:



- approach $(0,0)$ along the

- approach $(0,0)$ along the curve

Computing limits. Limit laws

Theorem 4.1 Let $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, $\lim\limits_{(x,y)\to(a,b)} g(x,y) = M$, c - constant

- $\lim\limits_{(x,y)\to(a,b)} c =$
- $\lim\limits_{(x,y)\to(a,b)} x =$
- $\lim\limits_{(x,y)\to(a,b)} y =$
- $\lim\limits_{(x,y)\to(a,b)} [f(x,y) \pm g(x,y)] =$
- $\lim\limits_{(x,y)\to(a,b)} [f(x,y)g(x,y)] =$
- , $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} =$
- $\lim\limits_{(x,y)\to(a,b)} [c\, f(x,y)] =$
- $\lim\limits_{(x,y)\to(a,b)} [f(x,y)]^n =$
- $\lim\limits_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} =$

Computing limits. Limit laws

Examples

$$\lim_{(x,y)\to(1,2)} \frac{\sqrt{3x-y}}{(x^2+xy+y^3)^2}$$

$$= \frac{\lim_{(x,y)\to(1,2)} \sqrt{3x-y}}{\lim_{(x,y)\to(1,2)} (x^2+xy+y^3)^2}$$

$$= \frac{\sqrt{\lim_{(x,y)\to(1,2)} [3x-y]}}{\left(\lim_{(x,y)\to(1,2)} [x^2+xy+y^3]\right)^2}$$

$$= \frac{\sqrt{3\lim_{(x,y)\to(1,2)} x - \lim_{(x,y)\to(1,2)} y}}{\left(\left(\lim_{(x,y)\to(1,2)} x\right)^2 + \left(\lim_{(x,y)\to(1,2)} x\right)\left(\lim_{(x,y)\to(1,2)} y\right) + \left(\lim_{(x,y)\to(1,2)} y\right)^3\right)^2}$$

$$= \frac{\sqrt{3\cdot 1 - 2}}{(1^2 + 1\cdot 2 + 2^3)^2} = \frac{1}{11^2} = \frac{1}{121}$$

Continuity of functions of two variables

Def. A function $f(x,y)$ is continuous at a point (a,b) if

(i)

(ii)

(iii)

Properties

1. If $f(x,y)$ and $g(x,y)$ are continuous at (x_0, y_0), then

   is continuous at (x_0, y_0)

2. If $\varphi(x)$ is continuous at x_0 and $\psi(y)$ is continuous at y_0,

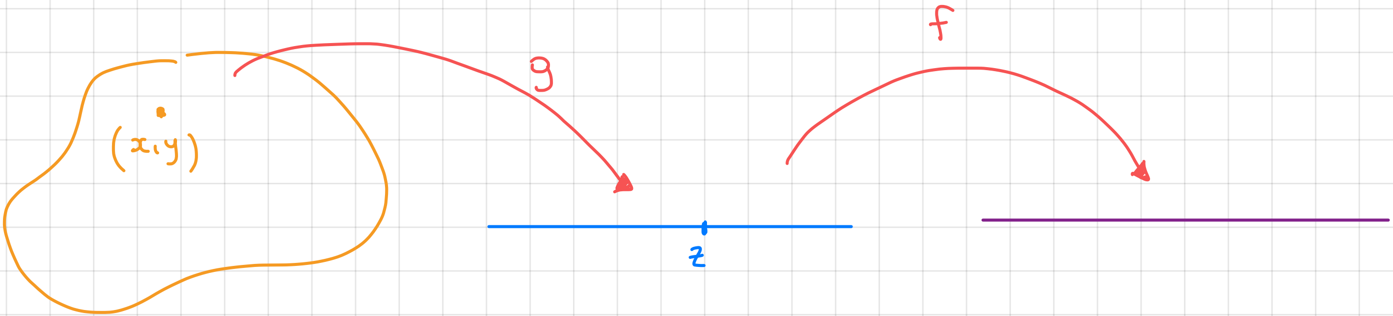
   then is continuous at (x_0, y_0)

Continuity of functions of two variables

Properties (cont.)

3. If $g(x,y)$ is continuous at (x_0, y_0), and $f(z)$ is continuous at $z_0 := g(x_0, y_0)$, then

is continuous at (x_0, y_0)



Continuity of functions of two variables

Example $\dfrac{\sqrt{3x-y}}{(x^2+xy+y^3)^2}$:

Similarly,

so

Take $(x_0, y_0) = (1, 2)$.