MATH 10C: Calculus III (Lecture B00)

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Today: Calculus of vector-valued functions. Tangent lines

Next: Strang 3.4

Week 4:

- homework 4 (due Friday, October 27)

Derivatives of vector-valued functions

The derivative of a vector-valued function $\vec{r}$ is

$$\vec{r}\,'(t) =$$

provided that the limit exists. If $\vec{r}\,'(t)$ exists, we say that $\vec{r}$ is   If $\vec{r}$ is differentiable at every point t from the interval (a,b), we say that $\vec{r}$ is differentiable on (a,b).

Notice that if $\vec{r}(t) = \langle r_1(t), r_2(t), r_3(t) \rangle$, then

$$\vec{r}\,'(t) =$$

$$=$$

Calculus of vector-valued functions

Example Let $\vec{r}(t) = \langle \sin t, e^{2t}, t^2 - 4t + 2 \rangle$

Then

Summary

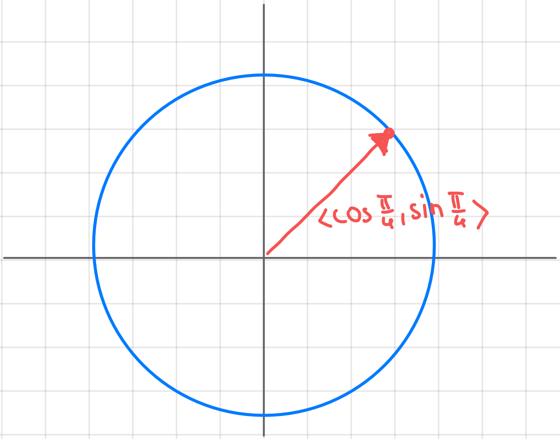
Calculus concepts (limit, continuity, derivative) are applied to vector-valued functions componentwise (apply to each component separately).

If $\vec{r}(t)$ represents the position of some object, then

- $\vec{r}\,'(t)$ is the velocity of this object ($\|\vec{r}\,'(t)\|$ is speed)
- $\vec{r}\,''(t)$ is the acceleration of the object

Tangent vectors. Tangent lines

Let $\vec{r}(t)$ be a vector-valued function. Suppose that $\vec{r}$ is differentiable at t_0.



$\vec{r}(t) = \langle \cos t, \sin t \rangle$

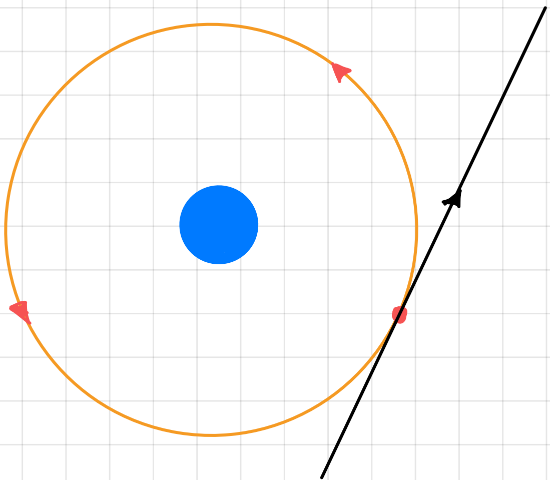
Then vector $\vec{r}\,'(t)$ is

The tangent line to $\vec{r}$ at t_0 is the line given by the vector equation

Tangent vectors. Tangent lines

The tangent line $\vec{\ell}(t)$ to $\vec{r}(t)$ at t_0 has the

Example Imagine satellite orbiting a planet.



If the planet disappears at time t_0, then

Tangent vectors. Tangent lines

<u>Example</u> Let $\vec{r}(t) = \langle t^2-2, e^{3t}, t \rangle$

Find the tangent line to $\vec{r}(t)$ at $t_0 = 1$.

First, find the tangent vector at $t_0 = 1$

Next, find the position at $t_0 = 1$

Finally, we can write the equation for the tangent line

<u>Definition</u> We call the principal unit tantent vector to $\vec{r}$ at t. (provided $\|\vec{r}'(t)\| \neq 0$)

Integrals of vector-valued functions

Integration of vector-valued functions is done

if $\vec{r}(t) = \langle r_1(t), r_2(t), r_3(t) \rangle$, then

and if $a < b$

Example $\bullet \int \langle \sin t, t^2 + 2t, e^{2t} \rangle dt =$

$=$

$\bullet \int_0^2 \left(\sin t \cdot \vec{i} + (t^2 + 2t) \cdot \vec{j} + e^{2t} \vec{k} \right) dt =$

$=$

Integrals of vector-valued functions

Fundamental theorem of calculus

Let $\vec{f} : [a,b] \to R^3$ be a continuous vector-valued function.
Let $\vec{F} : [a,b] \to \mathbb{R}^3$ be such that $\vec{F}' = \vec{f}$ ($\vec{F}$ is antiderivative of $\vec{f}$). Then

In particular,

- if $\vec{v}(t)$ is the velocity vector, $\vec{r}(t)$ is the position, then

  gives the

- if $\vec{a}(t)$ is the acceleration, then

Properties of derivatives of vector-valued functions

Thm 3.3. Let $\vec{r}(t)$ and $\vec{u}(t)$ be differentiable vector-valued functions, let $f(t)$ be a differentiable scalar function, let c be a scalar.

(i) $\frac{d}{dt}[c\vec{r}(t)] =$ (scalar multiple)

(ii) $\frac{d}{dt}[\vec{r}(t) \pm \vec{u}(t)] =$ (sum and difference)

(iii) $\frac{d}{dt}[f(t)\vec{r}(t)] =$ (product with scalar function)

(iv) $\frac{d}{dt}[\vec{r}(t) \cdot \vec{u}(t)] =$ (dot product)

(v) $\frac{d}{dt}[\vec{r}(t) \times \vec{u}(t)] =$ (cross product)

(vi) $\frac{d}{dt}[\vec{r}(f(t))] =$ (chain rule)

Properties of derivatives of vector-valued functions

(vii) If $\vec{r}(t) \cdot \vec{r}(t) = c$, then

Proof (iv) $\frac{d}{dt}[\vec{r}(t) \cdot \vec{u}(t)]$

$=$

$=$

$=$

(vii) $\frac{d}{dt}[\vec{r}(t) \cdot \vec{r}(t)]$

This means that if $\|\vec{r}(t)\|$ is constant, then

Motion in space

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is the position of the particle at time t, then

- $\vec{v}(t) = \vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$ is the velocity, and
- $\vec{a}(t) = \vec{r}\,''(t) = \langle x''(t), y''(t), z''(t) \rangle$ is the acceleration, and
- $v(t) = \|\vec{v}(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$ is the speed

Example: Projectile motion

Consider an object moving with initial velocity $\vec{v}_0$ but with no forces acting on it other than gravity (ignore the effect of air resistance).

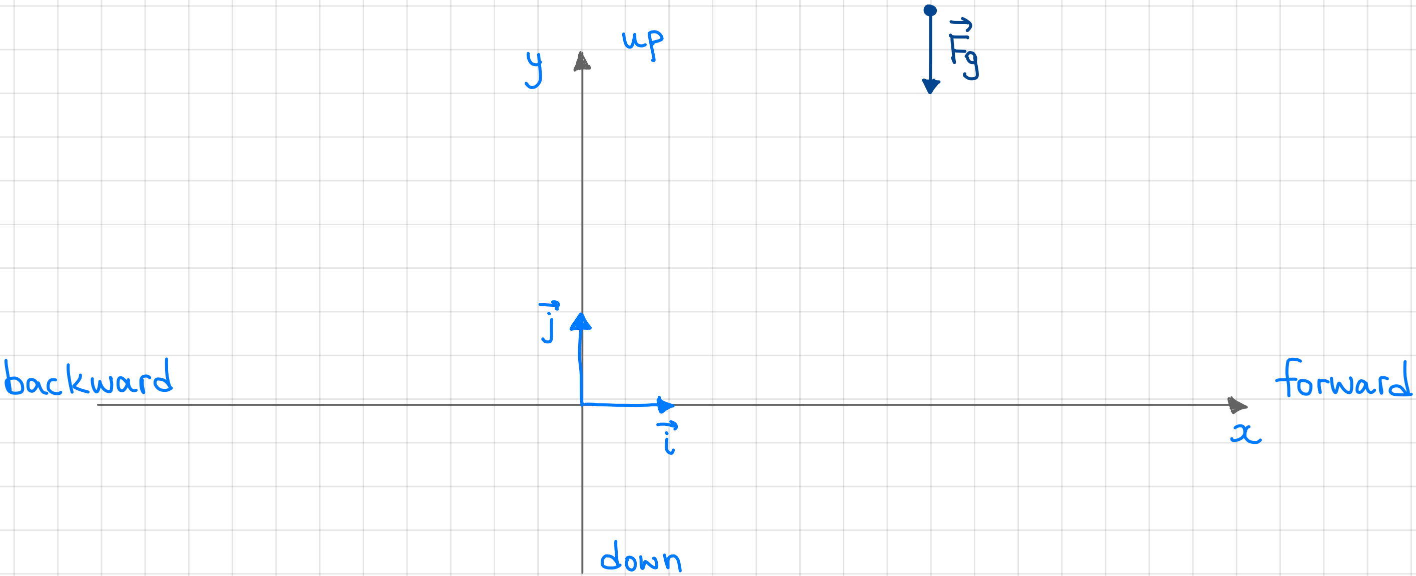
Newton's second law: $\vec{F} = m\vec{a}$, where m = mass of the object

Earth's gravity: $\|\vec{F}_g\| = mg$, where $g \approx 9.8 \ m/s^2$

Projectile motion

Fix the coordinate system:

$\vec{F}_g = -mg\,\vec{j} = \langle 0, -mg, 0 \rangle$



Newton's second law: $\vec{F} = m\,\vec{a}$

Earth's gravity: $\vec{F}_g = -mg\,\vec{j}$

Earth's gravity is the only force acting on the object

$\vec{F} = \vec{F}_g$

Projectile motion

$$\vec{F}(t) = \vec{F}_g : \quad m\vec{a}(t) = -mg\cdot\vec{j}$$

$$\vec{a}(t) = -g\cdot\vec{j} \quad (\text{constant acceleration})$$

Since $\vec{a}(t) = \vec{v}'(t)$, we have $\vec{v}'(t) = -g\cdot\vec{j}$.

Take antiderivative: $\vec{v}(t) = \int -g\,dt \cdot \vec{j} + \vec{c}_1 = -gt\cdot\vec{j} + \vec{c}_1$

Determine $\vec{c}_1$ by taking $\vec{v}(0) = \vec{v}_0$ (initial velocity):

$$\vec{v}(0) = -g\cdot 0\cdot\vec{j} + \vec{c}_1 = \vec{c}_1 = \vec{v}_0$$

This gives the velocity of the object:

$$\vec{v}(t) = -g\cdot t\cdot\vec{j} + \vec{v}_0.$$

Similarly, $\vec{v}(t) = \vec{r}'(t)$. By taking the antiderivative and $\vec{r}(0) = \vec{r}_0$

$$\vec{r}(t) = \int \vec{v}(t)\,dt + \vec{c}_0 = \int(-g\cdot t\cdot\vec{j} + \vec{v}_0)\,dt + \vec{c}_0 = -gt^2\vec{j} + t\vec{v}_0 + \vec{c}_0$$

$$\vec{r}(0) = \vec{c}_0 = \vec{r}_0, \quad \text{so} \quad \vec{r}(t) = -gt^2\cdot\vec{j} + t\vec{v}_0 + \vec{r}_0$$

Projectile motion

A projectile is shot by a howitzer with initial speed 800 m/s on a flat terrain. Determine the max distance the projectile can cover before hitting the ground.

Since the initial speed is given, the initial velocity can be determined by the angle: $\vec{V}_0 = 800 \langle \cos\theta, \sin\theta \rangle$

Equation of the trajectory: $\vec{r}(t) = -10 \cdot t^2 \vec{j} + 800t\cos\theta\,\vec{i} + 800t\sin\theta\,\vec{j}$

Hitting the ground: second component or $\vec{r}(t)$ is 0: $\left(-10t^2 + 800t\sin\theta\right) = 0$

$t\left(-10t + 800\sin\theta\right) = 0$, so $t_h = \dfrac{800 \cdot \sin\theta}{10} = 80 \cdot \sin\theta$. The position of the hit is

$\vec{r}(t_h) = 0 \cdot \vec{j} + 800 \cdot 80 \cdot \sin\theta \cdot \cos\theta \cdot \vec{i} = 64000 \cdot \frac{1}{2}\sin(2\theta)$. Maximum is achieved when $\sin(2\theta) = 1$, i.e., $2\theta = \frac{\pi}{2}$, $\theta = \frac{\pi}{4} = 45^\circ$. Max distance is 32 km.