MATH 285: Stochastic Processes

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Today: Periodic, aperiodic, reducible, irreducible Markov chains with finite state space

- Homework 2 is due on Friday, January 21 11:59 PM

Stationary distribution and long-run behavior

Prop. 7.1 Let (X_n) be a MC with finite state space S. Suppose that there exists $n_0 \in \mathbb{N}$ s.t $[P^{n_0}]_{ij} > 0$ for all $i,j \in S$. Then for each j, $\pi(j)$ is equal to the asymptotic expected fraction of time the chain spends in state j, i.e.,

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n+1}\sum_{k=0}^{n} \mathbb{1}_{\{X_k=j\}}\right] = \pi(j)$$

Proof.

$$\mathbb{E}\left[\frac{1}{n+1}\sum_{k=0}^{n} \mathbb{1}_{\{X_k=j\}}\right] = \frac{1}{n+1}\sum_{k=0}^{n} \mathbb{P}[X_k=j] = \frac{1}{n+1}\sum_{k=0}^{n}\sum_{i\in S} \mathbb{P}[X_k=j \mid X_0=i]\,\mathbb{P}[X_0=i]$$

$$= \frac{1}{n+1}\sum_{k=0}^{n} [\pi_0 P^k]_j$$

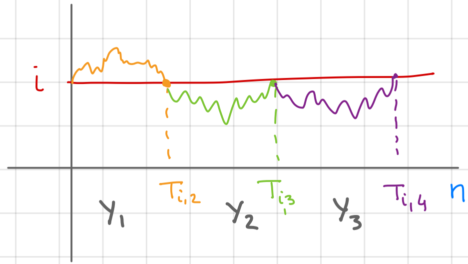
By Cor. 6.6, $[\pi_0 P^k]_j \to \pi(j)$, $k\to\infty$, for all $j \in S$ and π_0.

Therefore, $\frac{1}{n+1}\sum_{k=0}^{n} [\pi_0 P^k]_j \to \pi(j)$ ∎ $\left[\text{if } a_n \to a,\ n\to\infty,\ \text{then } \frac{1}{n}\sum_{k=1}^{n} a_k \xrightarrow[n\to\infty]{} a\right]$

Stationary distribution and expected return times

Recall that $T_{i,k}$ denotes the time of the k-th visit to state i.

$T_{i,k+1} =$



$T_{i,k}$ are stopping times. Denote

$Y_k =$, $k = 1, 2, \ldots$

Then by the strong Markov property

$\{Y_k\}_{k=1}^{\infty}$ is a collection of i.i.d. random variables

$Y_k \sim$. Notice that $\sum_{k=1}^{m} Y_k = \sum_{k=1}^{m} T_{i,k+1} - T_{i,k} =$

$\frac{1}{m} T_{i,m+1} =$, $m \to \infty$, so $T_{i,m+1} \approx$

Take m large, and let $n = m\,\mathbb{E}[T_i]$. Then

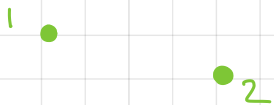
so $\sum_{k=0}^{n} \mathbb{1}_{\{X_k = i\}}$. Then $\frac{m+1}{n} \approx$

Periodic and aperiodic chains

Let (X_n) be a MC with state space S and transition probability $p(i,j)$.

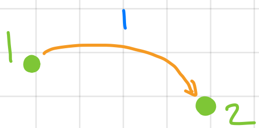
__Def__. For $i \in S$, denote $J_i :=$. We call

$d(i) :=$

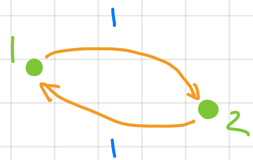


$J_1 =$

$d(1) =$

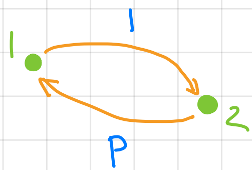


$J_1 =$



$J_1 =$

$d(1) =$



$p \in (0,1)$

$J_1 =$

$d(1) =$

__Def__ If $d(i) = 1$ for all $i \in S$, then (X_n) is called

Periodic and aperiodic chains

Lemma 7.2 If P is the transition matrix for an irreducible Markov chain, then for all states i,j.

Proof Fix $i \in S$.

(1) If $m, n \in J_i$, then

(2) Let $d = d(i)$. Then (definition of $d(i)$)

Take $j \neq i$.

(3) P irreducible $\Rightarrow \exists\, m, n$ s.t. $p_m(i,j) > 0$, $p_n(j,i) > 0$.

$\Rightarrow p_{m+n}(i,i) > 0 \Rightarrow \qquad \overset{(2)}{\Rightarrow}$

(4) If $\ell \in J_j$, then $p_\ell(j,j) > 0$ and thus

$\Rightarrow \qquad \Rightarrow \qquad \Rightarrow$

$\Rightarrow d$ is a common divisor of $J_j \Rightarrow$

(5) Swap i and j: $\exists\, q_2 \in \mathbb{N}$ s.t. $d(i) = q_2\, d(j) \overset{(4)}{\Rightarrow} d(i) = d(j)$

RW on bipartite graphs

Example 7.3 Let $G=(V,E)$ be finite connected graph.

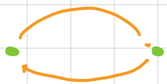
- SRW on G is irreducible (all vertices have the same period) — we call the common period the period of MC
- For any $i\sim j$ $p(i,j)>0$, $p(j,i)>0$, so $p_2(i,i)>0$, $2\in J_i$

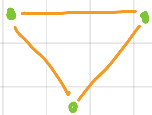
  $\Rightarrow$ $d(i)\leq 2$
- Period is 2 iff

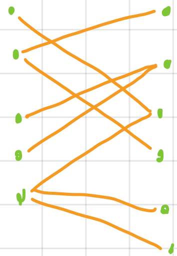
  $V=V_1\sqcup V_2$, $E\subset(V_1\times V_2\cup V_2\times V_1)$

  $V=\mathbb{Z}$, $V_1=$ even numbers

  $V_2=$ odd numbers







Irreducible aperiodic Markov chains

<u>Theorem 7.4</u> Let P be a transition matrix for a finite-state, irreducible, aperiodic Markov chain. Then there exists a unique stationary distribution π, $\pi = \pi P$, and for any initial probability distribution ν

$$\lim_{n\to\infty} \nu P^n = \pi$$

<u>Proof.</u> (1) By PF theorem, enough to show that there exists $n_0 > 0$ s.t. $\forall$ i,j . Fix $i,j \in S$

(2) $d(i) = 1$ (aperiodic) $\Rightarrow$ $\exists$ M_i s.t. J_i contains all $n \geq M_i$

$\hookrightarrow$ $P_n(i,i) > 0$

(3) irreducible $\Rightarrow$ $\exists$ m_{ij} s.t. $P_{m_{ij}}(i,j) > 0$

(2)+(3) :

Take $n_0 = \max\limits_{i,j} (M_i + m_{ij})$ $\Rightarrow$

∎

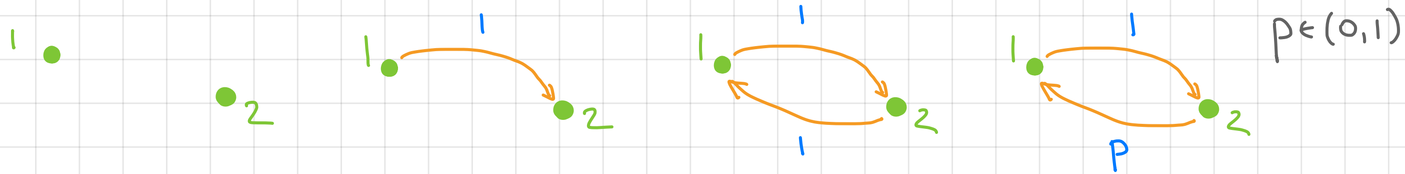
Reducible Markov chains

Not irreducible MC = reducible MC

Def 7.5 Let (X_n) be a MC with state space S.

We say that states i and j , denoted

if there exists $n, m \in \mathbb{N} \cup \{0\}$ s.t. and .



Lemma 7.6 Relation $\leftrightarrow$ on S is an equivalence relation.

(reflexivity, $i \leftrightarrow i$) $p_0(i,i) = 1$, so $i \leftrightarrow i$

(symmetry, $i \leftrightarrow j \Rightarrow j \leftrightarrow i$) Follows from Def 7.5

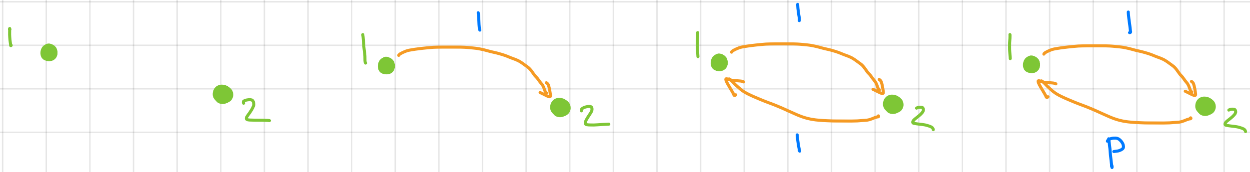
(transitivity, $i \leftrightarrow j, j \leftrightarrow k \Rightarrow i \leftrightarrow k$) $i \leftrightarrow j$: $p_n(i,j) > 0$, $p_m(j,i) > 0$

$j \leftrightarrow k$: $p_{n'}(j,k) > 0$, $p_{m'}(k,j) > 0$. Then

Communication classes

Equivalence relation $\leftrightarrow$ splits the state space into communication classes (sets of states that communicate with each other).

$p \in (0,1)$



MC is irreducible iff it consists of one communication class

Class properties: [proof as in Prop 4.8, Prop. 7.2]

- transience or recurrence: either all states in one class are transient (class) or all are recurrent (class)
- periodicity: all states in one class have the same period

Communication classes

Suppose i and j belong to different classes.

- If $p(i,j)>0$, then for all $n \in \mathbb{N}$ (otherwise $i \leftrightarrow j$.
- If $p(i,j)>0$ and $p_n(j,i)=0$ for all $n \in \mathbb{N}$, then $\mathbb{P}_i[X_n = i$ for infinitely many $n] \leq$ and thus i is transient
- Therefore, if i and j belong to different classes and i is recurrent, then (once in a recurrent class, MC stays there forever)

If we split the state space into communication classes, with R_ℓ denoting recurrent classes, then the transition matrix has the following form

General form of transition matrix with finite S

$$P = \left[\begin{array}{ccccc|c} P_1 & & & 0 & & 0 \\ & P_2 & & & & \\ & & P_3 & & & \\ & & & \ddots & & \\ & 0 & & & P_r & \\ \hline & & S & & & Q \end{array}\right]$$

P_ℓ submatrix for the recurrent class R_ℓ

P_ℓ is a stochastic matrix, we can consider it as a Markov chain on R_ℓ

$[S|Q]$ transition probabilities starting from transient states.

- If P_ℓ is aperiodic, then $P_\ell^n \to \begin{bmatrix} \pi^{(\ell)} \\ \vdots \\ \pi^{(\ell)} \end{bmatrix}$, $n \to \infty$
- What about transient states?
- What if P_ℓ is not aperiodic?

Transient states

Suppose there exists one transient class T

$$\begin{array}{r} \\ \\ \\ \\ \\ T\{ \end{array}\left[\begin{array}{ccccc|c} P_1 & & & & 0 & 0 \\ & P_2 & & & & \\ & & P_3 & & & \\ & & & \ddots & & \\ 0 & & & & P_r & \\ \hline \multicolumn{5}{c|}{S} & Q \end{array}\right]$$

- If $S = 0$ then T is recurrent
- If $S \neq 0$, then Q is substochastic,

  i.e., $\exists\ i \in T$ s.t. $\sum_{j \in T} Q < 1$
- If Q is substochastic, then for all eigenvalues λ of Q $|\lambda| < 1$

  $\Rightarrow Q^n \to 0,\ n \to \infty$, i.e. for $i, j \in T$ $\mathbb{P}_i[X_n = j] \to 0,\ n \to \infty$
- $I + Q + Q^2 + \cdots = I + VDV^{-1} + VD^2V^{-1} + \cdots = V(I + D + D^2 + \cdots)V^{-1}$ converges

For $i, j \in T$, $\mathbb{E}_i\left[\sum_{k=0}^{n} \mathbb{1}_{X_k = j}\right] =$

$=$

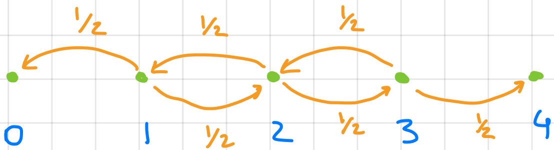
Transient states

Conclusion: if $T \subset S$ is a transient class, then $\forall\, i, j \in T$

$$\lim_{n \to \infty} \mathbb{P}_i [X_n = j] =$$

$$\mathbb{E}_i \Big[\sum_{k=0}^{n} \mathbb{1}_{\{X_n = j\}} \Big] = \quad \text{expected number of visits to } j \text{ starting from } i$$

Example 8.1



$$P = \begin{array}{c} \\ 0 \\ 4 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 4 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

$$Q = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \qquad (I - Q)^{-1} = \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ 1 & 2 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix}$$

Expected number of visits to 2 starting from 1 is 1

Expected number of steps before absorption starting from 1 is $\frac{3}{2} + 1 + \frac{1}{2} = 3$

Transient states

Recall, First step analysis for the mean hitting time

$$g_i = \mathbb{E}_i[\tau_A] = \begin{cases} 0, & i \in A \\ 1 + \sum_{j \in S} P(i,j)\, g_j, & i \notin A \end{cases}$$

$$\tau_A = \sum_{n=0}^{\infty} \mathbb{1}_{\{X_n \notin A\}}$$

Instead of adding 1 for each step, add 1 only when X_n visits j:

Denote $S \setminus A =: T$, and for $i, j \in T$ $g_{ij} =$

Then FSA $g_{ij} =$ if $i \in A$

$g_{ij} =$

$G = [g_{ij}]$, then

Transient states

Starting from T, in which class will (X_n) end up ?

$$\begin{array}{c} R_1 \\ R_2 \\ \\ \\ \\ T \end{array}\left[\begin{array}{ccccc|c} P_1 & & & 0 & & 0 \\ & P_2 & & & & \\ & & P_3 & & & \\ & 0 & & \ddots & & \\ & & & & P_r & \\ \hline & & S & & & Q \end{array}\right]$$

Collapse each R_e into one state r_e, keep transient states t_e, $T = \{t_e\}$

$(\tilde{X}_n)$ new MC on the reduced state space, and transition matrix $\tilde{P}$,

$$\tilde{P} = \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ \\ \\ \\ t_1 \\ t_2 \end{array}\left[\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \hline & & \tilde{S} & & & Q \\ & & & & & \end{array}\right]$$

with $\tilde{s}(t_i, r_j) =$

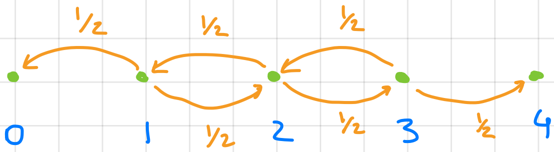
Denote $\tilde{A} = [\alpha(t_i, r_j)]$ with

$\alpha(t_i, r_j) :=$

Then $\tilde{A} =$

Transient states

Example 8.2



$$P = \begin{array}{c} \\ 0 \\ 4 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 4 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{array}\right] \end{array}$$

What is the probability that starting from a transient state i we end up in a recurrent state j?

Use $\tilde{A} =$ (nothing to collapse in this case)

$$\tilde{A} = \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ 1 & 2 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

- Expected transit times from i to j (think about j as absorbing)...