MATH 285: Stochastic Processes

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Today: Periodic, aperiodic, reducible, irreducible Markov chains with finite state space

- Homework 2 is due on Friday, January 21 11:59 PM

Stationary distribution and long-run behavior

Prop. 7.1 Let (X_n) be a MC with finite state space S.
Suppose that there exists $n_0 \in \mathbb{N}$ s.t $[P^{n_0}]_{ij} > 0$ for all $i, j \in S$.
Then for each j, $\pi(j)$ is equal to the asymptotic expected fraction of time the chain spends in state j, i.e.,

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n+1}\sum_{k=0}^{n} \mathbb{1}_{\{X_k=j\}}\right] = \pi(j)$$

Proof.

$$\mathbb{E}\left[\frac{1}{n+1}\sum_{k=0}^{n} \mathbb{1}_{\{X_k=j\}}\right] = \frac{1}{n+1}\sum_{k=0}^{n} \mathbb{P}[X_k=j] = \frac{1}{n+1}\sum_{k=0}^{n}\sum_{i\in S} \mathbb{P}[X_k=j \mid X_0=i]\,\mathbb{P}[X_0=i]$$

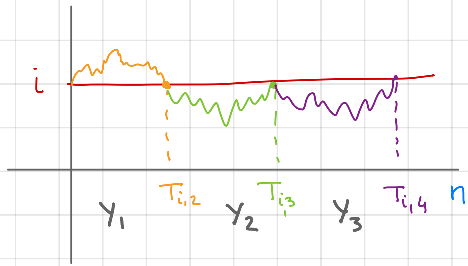
$$= \frac{1}{n+1}\sum_{k=0}^{n}\left[\pi_0 P^k\right]_j$$

By Cor. 6.6, $[\pi_0 P^k]_j \to \pi(j)$, $k\to\infty$, for all $j\in S$ and π_0.
Therefore, $\frac{1}{n+1}\sum_{k=0}^{n}[\pi_0 P^k]_j \to \pi(j)$ ■ $\left[\text{if } a_n \to a,\ n\to\infty,\ \text{then } \frac{1}{n}\sum_{k=1}^{n} a_k \xrightarrow[n\to\infty]{} a\right]$

Stationary distribution and expected return times

Recall that $T_{i,k}$ denotes the time of the k-th visit to state i.

$$T_{i,k+1} = \min\{n > T_{i,k} : X_n = i\}$$



$T_{i,k}$ are stopping times. Denote

$$Y_k = T_{i,k+1} - T_{i,k}, \quad k = 1, 2, \ldots$$

Then by the strong Markov property $\{Y_k\}_{k=1}^{\infty}$ is a collection of i.i.d. random variables

$Y_k \sim T_i = T_{i,2}$. Notice that $\sum_{k=1}^{m} Y_k = \sum_{k=1}^{m} T_{i,k+1} - T_{i,k} = T_{i,m+1}$

$$\frac{1}{m} T_{i,m+1} = \frac{1}{m} \sum_{k=1}^{m} Y_k \to \mathbb{E}[T_i], \quad m \to \infty, \text{ so } T_{i,m+1} \approx m \cdot \mathbb{E}[T_i]$$

Take m large, and let $n = m\,\mathbb{E}[T_i]$. Then $T_{i,m+1} \approx n$

so $\sum_{k=0}^{n} \mathbb{1}_{\{X_k = i\}} = m+1 \overset{\text{Prop 7.1}}{\approx} n\pi(i)$. Then $\frac{m+1}{n} \approx \pi(i) \approx \frac{1}{\mathbb{E}[T_i]}$

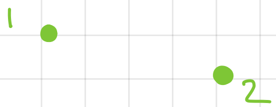
Periodic and aperiodic chains

Let (X_n) be a MC with state space S and transition probability $p(i,j)$.

<u>Def</u>. For $i \in S$, denote $J_i := \{n \geq 0 : p_n(i,i) > 0\}$. We call

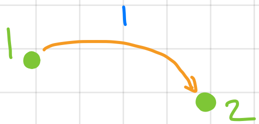
$d(i) :=$ greatest common divisor of J_i (period of i)

$p \in (0,1)$



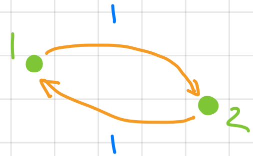
$J_1 = \{0, 1, 2, \dots\}$

$d(1) = 1$



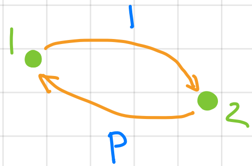
$J_1 = \{0\}$

no period



$J_1 = \{0, 2, 4, \dots\}$

$d(1) = 2$



$J_1 = \{0, 2, 3, 4, 5, \dots\}$

$d(1) = 1$

<u>Def</u> If $d(i) = 1$ for all $i \in S$, then (X_n) is called aperiodic

Periodic and aperiodic chains

Lemma 7.2 If P is the transition matrix for an irreducible Markov chain, then $d(i)=d(j)$ for all states i,j.

Proof. Fix $i \in S$.

(1) If $m,n \in J_i$, then $m+n \in J_i$

(2) Let $d=d(i)$. Then $J_i \subset \{0,d,2d,\dots\}$ (definition of $d(i)$)

Take $j \neq i$.

(3) P irreducible $\Rightarrow \exists\, m,n$ s.t. $p_m(i,j)>0$, $p_n(j,i)>0$.

$\Rightarrow p_{m+n}(i,i)>0 \Rightarrow m+n \in J_i \overset{(2)}{\Rightarrow} \exists\, k \in \mathbb{N} : m+n = kd$

(4) If $\ell \in J_j$, then $p_\ell(j,j)>0$ and thus $p_{m+\ell+n}(i,i)>0$

$\Rightarrow m+\ell+n \in J_i \Rightarrow \exists\, k' : \ell = k'd \Rightarrow \ell$ divisible by d

$\Rightarrow d$ is a common divisor of $J_j \Rightarrow \exists\, q_1 \in \mathbb{N}$ s.t. $d(j) = q_1 d(i)$

(5) Swap i and j: $\exists\, q_2 \in \mathbb{N}$ s.t. $d(i) = q_2 d(j) \overset{(4)}{\Rightarrow} d(i) = d(j)$ ∎

RW on bipartite graphs

Example 7.3 Let $G=(V,E)$ be finite connected graph.

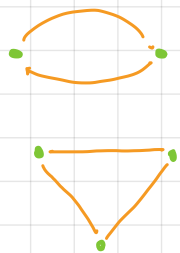
- SRW on G is irreducible (all vertices have the same period) — we call the common period the period of MC
- For any $i \sim j$ $p(i,j)>0$, $p(j,i)>0$, so $p_2(i,i)>0$, $2 \in J_i$

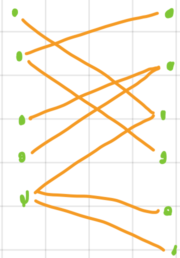
  $\Rightarrow$ $d(i) \leq 2$
- Period is 2 iff G is bipartite:

  $V = V_1 \sqcup V_2$, $E \subset (V_1 \times V_2 \cup V_2 \times V_1)$

  $V = \mathbb{Z}$, V_1 = even numbers

  V_2 = odd numbers





Irreducible aperiodic Markov chains

Theorem 7.4 Let P be a transition matrix for a finite-state, irreducible, aperiodic Markov chain. Then there exists a unique stationary distribution π, $\pi = \pi P$, and for any initial probability distribution ν

$$\lim_{n\to\infty} \nu P^n = \pi$$

Proof. (1) By PF theorem, enough to show that there exists $n_0 > 0$ s.t. $\forall\ i,j$ $[P^{n_0}]_{ij} > 0$. Fix $i,j \in S$

(2) $d(i) = 1$ (aperiodic) $\Rightarrow \exists\ M_i$ s.t. J_i contains all $n \geq M_i$

↳ $P_n(i,i) > 0$

(3) irreducible $\Rightarrow \exists\ m_{ij}$ s.t. $P_{m_{ij}}(i,j) > 0$

(2)+(3) : $\forall n \geq M_i + m_{ij}$ $\quad P_n(i,j) > 0$

Take $\quad n_0 = \max_{i,j} (M_i + m_{ij}) \Rightarrow \forall\ i,j \in S \quad P_{n_0}(i,j) > 0$

∎

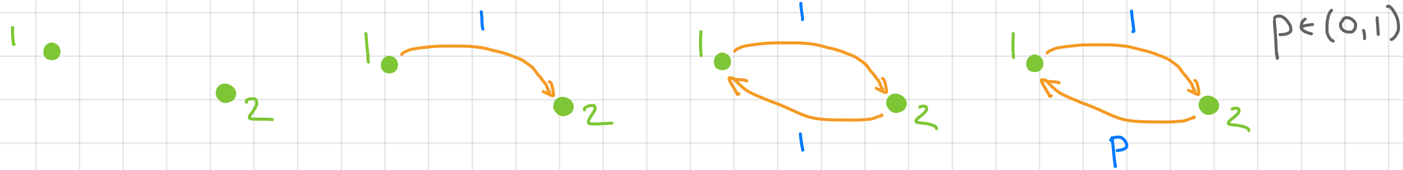
Reducible Markov chains

Not irreducible MC = reducible MC

Def 7.5 Let (X_n) be a MC with state space S.

We say that states i and j **communicate**, denoted $i \leftrightarrow j$

if there exist $n, m \in \mathbb{N} \cup \{0\}$ s.t. $p_n(i,j) > 0$ and $p_m(j,i) > 0$.



Lemma 7.6 Relation $\leftrightarrow$ on S is an equivalence relation.

(reflexivity, $i \leftrightarrow i$) $p_0(i,i) = 1$, so $i \leftrightarrow i$

(symmetry, $i \leftrightarrow j \Rightarrow j \leftrightarrow i$) Follows from Def 7.5

(transitivity, $i \leftrightarrow j, j \leftrightarrow k \Rightarrow i \leftrightarrow k$) $i \leftrightarrow j$: $p_n(i,j) > 0$, $p_m(j,i) > 0$

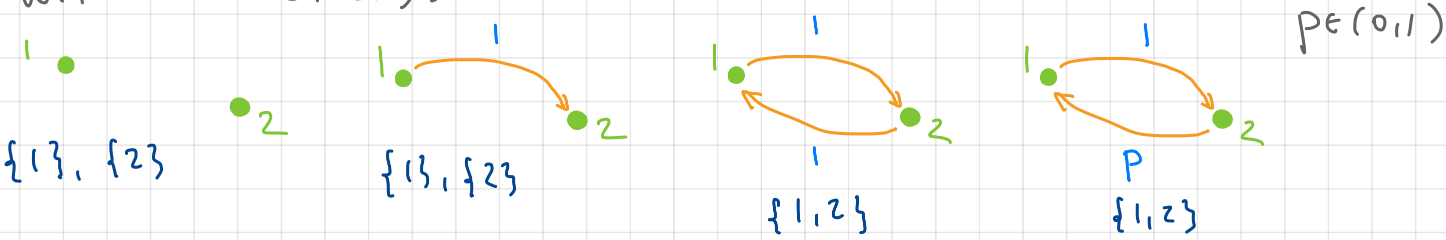
$j \leftrightarrow k$: $p_{n'}(j,k) > 0$, $p_{m'}(k,j) > 0$. Then $p_{n+n'}(i,k) > 0$

$p_{m'+m}(k,i) > 0$

∎

Communication classes

Equivalence relation $\leftrightarrow$ splits the state space into communication classes (sets of states that communicate with each other).



MC is irreducible iff it consists of one communication class

Class properties: [proof as in Prop 4.8, Prop. 7.2]

- transience or recurrence: either all states in one class are transient (class) or all are recurrent (class)
- periodicity: all states in one class have the same period

Communication classes

Suppose i and j belong to different classes.

- If $p(i,j) > 0$, then $p_n(j,i) = 0$ for all $n \in \mathbb{N}$ (otherwise $i \leftrightarrow j$.
- If $p(i,j) > 0$ and $p_n(j,i) = 0$ for all $n \in \mathbb{N}$, then $\mathbb{P}_i[X_n = i \text{ for infinitely many } n] \leq 1 - p(i,j) < 1$, and thus i is transient
- Therefore, if i and j belong to different classes and i is recurrent, then $p(i,j) = 0$ (once in a recurrent class, MC stays there forever)

If we split the state space into communication classes, with R_ℓ denoting recurrent classes, then the transition matrix has the following form

General form of transition matrix with finite S

$$P^n = \begin{array}{c|cccccc|c|}
R_1 & P_1^n & & & & & 0 & 0 \\
 & & P_2^n & & & & & \\
 & & & P_3^n & & & & \\
 & & & & \ddots & & & \\
R_\ell & 0 & & & & P_r^n & & \\
\hline
T & & & S_n & & & & Q^n
\end{array}$$

P_ℓ submatrix for the recurrent class R_ℓ

P_ℓ is a stochastic matrix, we can consider it as a Markov chain on R_ℓ

$[S|Q]$ transition probabilities starting from transient states.

- If P_ℓ is aperiodic, then $P_\ell^n \to \begin{bmatrix} \pi^{(\ell)} \\ \vdots \\ \pi^{(\ell)} \end{bmatrix}$, $n \to \infty$
- What about transient states?
- What if P_ℓ is not aperiodic?