MATH 285: Stochastic Processes

math-old.ucsd.edu/~ynemish/teaching/285

Today: Periodic, aperiodic, reducible, irreducible Markov chains

- Homework 2 is due on Friday, January 21 11:59 PM

First step analysis

Let (X_n) be a MC with state space S and transition matrix P.

Let $A \subset S$, $\tau_A = \min\{n \geq 0 : X_n \in A\}$, and denote
$h^A(i) := \mathbb{P}_i[\tau_A < \infty]$ (as in lecture 3 with $B = \emptyset$, so that $\tau_B = \infty$)

Then (lecture 2) $h^A(i)$ satisfies the system of linear equations

$$(*) \quad \begin{cases} h^A(i) = 1 & \text{if } i \in A \\ h^A(i) = \sum_{j \in S} p(i,j)\, h^A(j) & \text{if } i \notin A \end{cases}$$

The solution may be not unique.

<u>Theorem 7.0</u> The vector of hitting probabilities $(h^A(i), i \in S)$ is the

(Minimal: if $(x(i), i \in S)$ satisfies $(*)$ and $x(i) \geq 0$ $\forall i$, then $\underset{\forall i}{x(i) \geq h^A(i)}$)

First step analysis

Proof of minimality: Let $(x(i), i \in S)$ be a nonnegative solution to $(*)$. Then $x(i) = 1$ for all $i \in A$ (so $x(i) \geq h^A(i)$)

For all $i \notin A$

$$x(i) = \sum_{j \in S} p(i,j) x_j =$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

$\Rightarrow \forall i \notin A \quad x(i) \geq \mathbb{P}_i[\tau_A \leq n] \Rightarrow$

First step analysis

Denote $g^A(i) := \mathbb{E}_i[\tau_A]$ (mean hitting / absorption time)

Theorem 7.0' The vector of mean hitting times $(g^A(i), i \in S)$ is the the   to the system of linear equations

$$\begin{cases} g(i) = 1 + \sum_{j \in S} p(i,j)\, g(j) & \text{if } i \notin A \\ g(i) = 0 & \text{if } i \in A \end{cases}$$

Proof: Exercise.

Stationary distribution

Stationary distribution

$$\pi = \pi P$$

Q 1 : Existence of the stationary distribution

Q 2 : Uniqueness of the stationary distribution

Q 3 : Convergence to the stationary distribution

General Markov chain with finite state space

Let (X_n) be a MC with finite state space S.

Suppose that $\pi = P\pi$, $P = Q D Q^{-1}$ such that

$$Q = \begin{bmatrix} 1 & \\ 1 & * \\ \vdots & \\ 1 & \end{bmatrix} \qquad Q^{-1} = \begin{bmatrix} \pi \\ \\ * \\ \\ \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & M \\ 0 & \end{bmatrix}, \quad \lim_{n\to\infty} M^n = 0 \ (**)$$

Then $\displaystyle\lim_{n\to\infty} P^n = \lim_{n\to\infty} Q D^n Q^{-1} = \begin{bmatrix} 1 & \\ 1 & * \\ \vdots & \\ 1 & \end{bmatrix} \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & 0 \\ 0 & \end{bmatrix} \begin{bmatrix} \pi \\ * \\ \\ \end{bmatrix} = \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix}$

Enough to have the following: (use Jordan normal form)

1) 1 is a simple eigenvalue (1 is always an eigenvalue since $(P\mathbb{1})_i = \sum_j p(i,j) = 1$, so $P\mathbb{1} = \mathbb{1}$, $\mathbb{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is an e.$\vec{v}$.)

2) There is a left eigenvector of 1 with all nonnegative entries

3) If λ is an eigenvalue of P and $\lambda = 1$, then $|\lambda| < 1$

Perron-Frobenius theorem

Theorem 6.5 Let M be an $N \times N$ matrix all of whose entries are strictly positive. Then

Moreover,

eigenspace contains a vector with

. Finally,

Proof. No proof ■

Let P be a stochastic matrix with all strictly positive entries.

Then , therefore 1 is the PF eigenvalue:

with (left) eigenvector π with If (X_n) is

a MC with transition matrix P, then

Perron - Frobenius Theorem

Enough if $\exists\, n_0 > 0$ s.t. all entries of P^{n_0} are strictly positive.

Corollary 6.6 Let P be a stochastic matrix s.t. there exists $n_0 \in \mathbb{N}$ for which

Then there exists a unique stationary distribution

and for any distribution ν.

Proof. Use the fact that if $\vec{v}P = \lambda \vec{v}$, then

so P^n has the same eigenvectors as P, and eigenvalues are n-th powers of eigenvalues of P. By PF thm, 1 is ev of P^n with $e\vec{v}s$ π and $\mathbb{1}$, therefore

If λ is ev of P and $\lambda \neq 1$, . By PF , therefore . We conclude that P satisfies

Stationary distribution and long-run behavior

Prop. 7.1 Let (X_n) be a MC with finite state space S.
Suppose that there exists $n_0 \in \mathbb{N}$ s.t $[P^{n_0}]_{ij} > 0$ for all $i,j \in S$
Then for each j, $\pi(j)$ is equal to the

Proof.

$$\mathbb{E}\left[\frac{1}{n+1} \sum_{k=0}^{n} \mathbb{1}_{\{X_k = j\}} \right] =$$

$$=$$

By Cor. 6.6, , for all $j \in S$ and π_0.

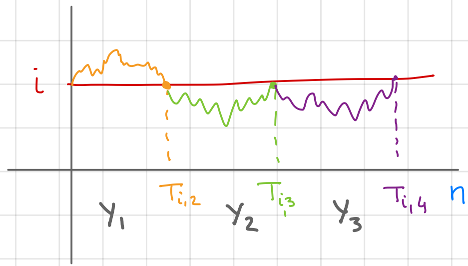
Therefore, $\left[\text{if } a_n \to a,\ n \to \infty,\ \text{then } \frac{1}{n} \sum_{k=1}^{n} a_k \xrightarrow[n\to\infty]{} a \right]$

Stationary distribution and expected return times

Recall that $T_{i,k}$ denotes the time of the k-th visit to state i.

$$T_{i,k+1} =$$

$T_{i,k}$ are stopping times. Denote



$$Y_k = \qquad\qquad , \quad k = 1, 2, \ldots$$

Then by the strong Markov property

$\{Y_k\}_{k=1}^{\infty}$ is a collection of i.i.d. random variables

$Y_k \sim \qquad\qquad$. Notice that $\sum_{k=1}^{m} Y_k = \sum_{k=1}^{m} T_{i,k+1} - T_{i,k} =$

$\frac{1}{m} T_{i,m+1} = \qquad\qquad , \quad m \to \infty, \quad$ so $\quad T_{i,m+1} \approx$

Take m large, and let $n = m\,\mathbb{E}[T_i]$. Then

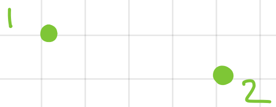
so $\sum_{k=0}^{n} \mathbb{1}_{\{X_k = i\}} \qquad\qquad$. Then $\frac{m+1}{n} \approx$

Periodic and aperiodic chains

Let (X_n) be a MC with state space S and transition probability $p(i,j)$.

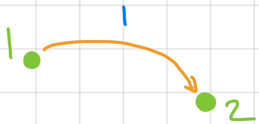
<u>Def</u>. For $i \in S$, denote $J_i :=$. We call

$d(i) :=$

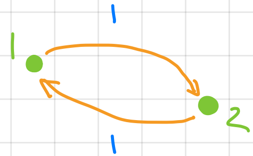


$J_1 =$

$d(1) =$

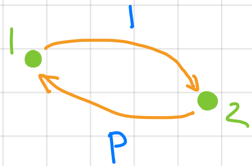


$J_1 =$



$J_1 =$

$d(1) =$



$p \in (0,1)$

$J_1 =$

$d(1) =$

<u>Def</u> If $d(i) = 1$ for all $i \in S$, then (X_n) is called

Periodic and aperiodic chains

Lemma 7.2 If P is the transition matrix for an irredusible Markov chain, then for all states i, j.

Proof. Fix $i \in S$.

(1) If $m, n \in J_i$, then

(2) Let $d = d(i)$. Then (definition of $d(i)$)

Take $j \neq i$.

(3) P irreducible $\Rightarrow \exists\, m, n$ s.t. $p_m(i,j) > 0$, $p_n(j,i) > 0$.

$\Rightarrow p_{m+n}(i,i) > 0 \Rightarrow \quad \overset{(2)}{\Rightarrow}$

(4) If $l \in J_j$, then $p_l(j,j) > 0$ and thus

$\Rightarrow \quad \Rightarrow \quad \Rightarrow$

$\Rightarrow$ d is a common divisor of $J_j \Rightarrow$

(5) Swap i and j: $\exists\, q_2 \in \mathbb{N}$ s.t. $d(i) = q_2\, d(j) \overset{(4)}{\Rightarrow} d(i) = d(j)$

RW on bipartite graphs

Example 7.3 Let $G=(V,E)$ be finite connected graph.

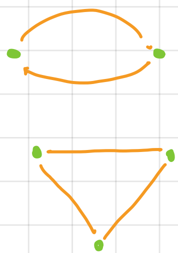
- SRW on G is irreducible (all vertices have the same period) — we call the common period the period of MC
- For any $i \sim j$ $p(i,j)>0$, $p(j,i)>0$, so $p_2(i,i)>0$, $2 \in J_i$

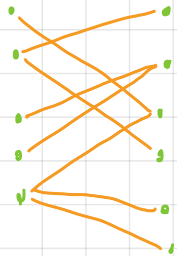
  $\Rightarrow d(i) \leq 2$
- Period is 2 iff

  $V = V_1 \sqcup V_2$, $E \subset (V_1 \times V_2 \cup V_2 \times V_1)$

  $V = \mathbb{Z}$, V_1 = even numbers

  V_2 = odd numbers





Irreducible aperiodic Markov chains

Theorem 7.4 Let P be a transition matrix for a finite-state, irreducible, aperiodic Markov chain. Then there exists a unique stationary distribution π, $\pi = \pi P$, and for any initial probability distribution ν

$$\lim_{n \to \infty} \nu P^n = \pi$$

Proof. (1) By PF theorem, enough to show that there exists $n_0 > 0$ s.t. $\forall\, i,j$ Fix $i,j \in S$

(2) $d(i) = 1$ (aperiodic) $\Rightarrow \exists\, M_i$ s.t. J_i contains all $n \geq M_i$

↳ $p_n(i,i) > 0$

(3) irreducible $\Rightarrow \exists\, m_{ij}$ s.t. $p_{m_{ij}}(i,j) > 0$

(2) + (3) :

Take $n_0 = \max_{i,j} (M_i + m_{ij}) \Rightarrow$

∎

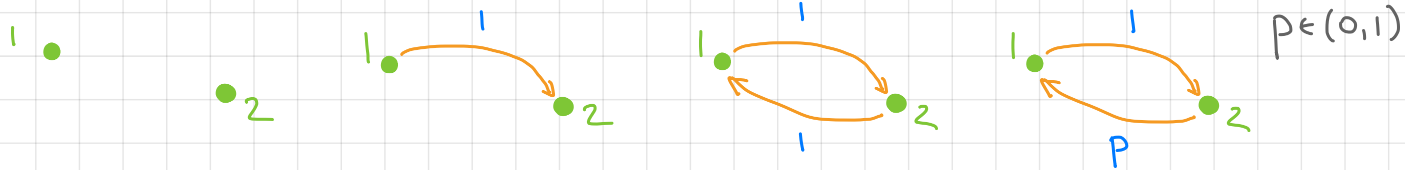
Reducible Markov chains

Not irreducible MC = reducible MC

Def 7.5 Let (X_n) be a MC with state space S.

We say that states i and j , denoted ,

if there exists $n, m \in \mathbb{N} \cup \{0\}$ s.t. and .



Lemma 7.6 Relation $\leftrightarrow$ on S is an equivalence relation.

(reflexivity, $i \leftrightarrow i$) $p_0(i,i) = 1$, so $i \leftrightarrow i$

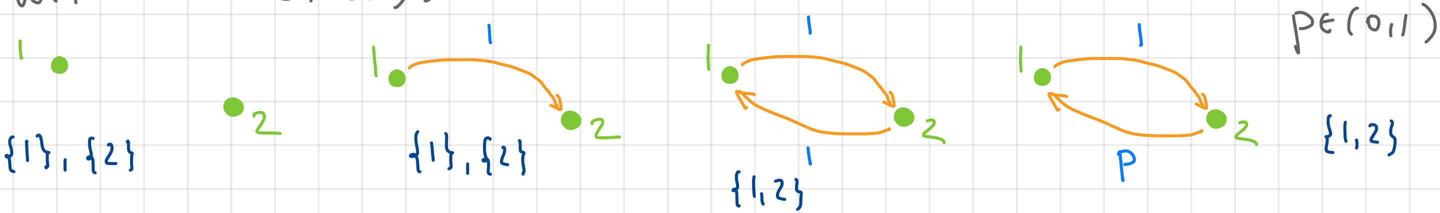
(symmetry, $i \leftrightarrow j \Rightarrow j \leftrightarrow i$) Follows from Def 7.5

(transitivity, $i \leftrightarrow j, j \leftrightarrow k \Rightarrow i \leftrightarrow k$) $i \leftrightarrow j$: $p_n(i,j) > 0$, $p_m(j,i) > 0$

$j \leftrightarrow k$: $p_{n'}(j,k) > 0$, $p_{m'}(k,j) > 0$. Then

Communication classes

Equivalence relation $\leftrightarrow$ splits the state space into communication classes (sets of states that communicate with each other).



MC is irreducible iff it consists of one communication class

Class properties:

- transience or recurrence: either all states in one class are transient (class) or all are recurrent (class)
- periodicity: all states in one class have the same period

Communication classes

Suppose i and j belong to different classes.

- If $p(i,j)>0$, then for all $n \in \mathbb{N}$ (otherwise $i \leftrightarrow j$.
- If $p(i,j)>0$ and $p_n(j,i)=0$ for all $n \in \mathbb{N}$, then $\mathbb{P}_i[X_n = i \text{ for infinitely many } n] \leq$, and thus i is transient
- Therefore, if i and j belong to different classes and i is recurrent, then (once in a recurrent class, MC stays there forever)

If we split the state space into communication classes, with R_e denoting recurrent classes, then the transition matrix has the following form

Communication classes

$$P = \left[\begin{array}{cccc|c} P_1 & & & 0 & 0 \\ & P_2 & & & \\ & & P_3 & & \\ 0 & & & \ddots \; P_r & \\ \hline & S & & & Q \end{array}\right]$$

P_i submatrix for the recurrent class R_i

P_i is a stochastic matrix, we can consider it as a Markov chain on R_i